An example on calculating a double integral by substitution

EXAMPLE. Let D be the set bounded by the curves given in $\mathbb{R}^2$ by the equations

$$y = x^2, y = x^2 + 1, y = 2 - x^2 \text{ and } y = 3 - x^2.$$

The region is depicted in the Figure. Make the substitutions

$$u = y - x^2 \quad \text{and} \quad v = y + x^2.$$

We have $v \geq u$. Note that

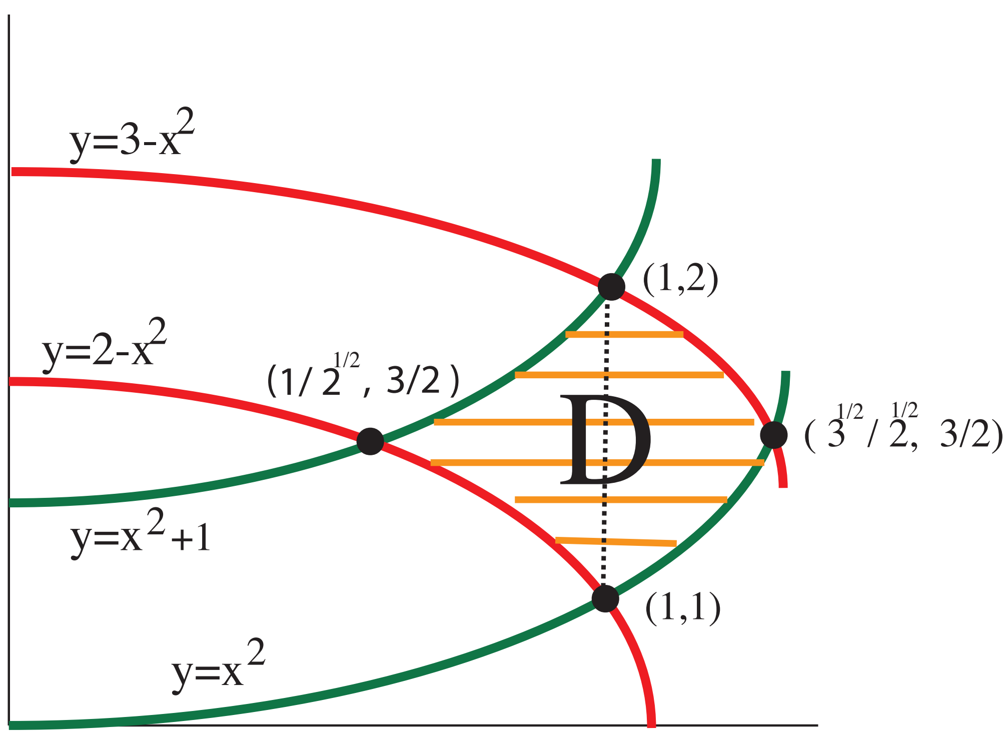
$$x = \sqrt{\frac{-u+v}{2}} \text{ and } y = \frac{u+v}{2}.$$

Then,

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} -2x & 1 \\ 2x & 1 \end{vmatrix} = -4x,$$

so that

$$\text{“}dudv = 4xdxdy\text{”}.$$



Under the substitution, that is under the function

$$(x, y) \longmapsto (y - x^2, y + x^2),$$

D changes into the region E that is bounded by the curves whose equations are

$$u = 0, u = 1, v = 2 \text{ and } v = 3.$$

Thus, E is a rectangle, $E = [0,1] \times [2,3]$ and we have

$$\begin{aligned}
\int\int_D x^2 dxdy &= \int\int_D xx dxdy \\
&= \frac{1}{4}\int\int_E \sqrt{\frac{-u+v}{2}} dudv \\
&= \frac{1}{4}\int_2^3 \left(\int_0^1 \sqrt{\frac{-u+v}{2}} du\right) dv \\
&= \frac{1}{4\sqrt{2}}\int_2^3 \left(\left[-\frac{2}{3}(-u+v)^{3/2}\right]_0^1\right) dv \\
&= -\frac{1}{6\sqrt{2}}\int_2^3 \left[(v-1)^{3/2} - v^{3/2}\right] dv \\
&= -\frac{1}{6\sqrt{2}} \cdot \frac{2}{5}\left[(v-1)^{5/2} - v^{5/2}\right]_2^3 \\
&= -\frac{1}{15\sqrt{2}}\left[2^{5/2} - 3^{5/2} - 1 + 2^{5/2}\right] \\
&= -\frac{1}{15\sqrt{2}}\left[8\sqrt{2} - 9\sqrt{3} - 1\right] \\
&= -\frac{1}{15\sqrt{2}}\left[-8\sqrt{2} + 9\sqrt{3} + 1\right] \\
&= \frac{1}{15}\left[9\sqrt{\frac{3}{2}} + \frac{1}{\sqrt{2}} - 8\right].
\end{aligned}$$

Note that the integral could also be worked out without substitution, by splitting D into two parts, indicated by the vertical dotted line. Then the same integral becomes

$$\begin{aligned}
&\int\int_D x^2 dxdy \\
&= \int_{1/\sqrt{2}}^1 \left(\int_{2-x^2}^{x^2+1} x^2 dy\right) dx + \int_1^{\sqrt{3/2}} \left(\int_{x^2}^{3-x^2} x^2 dy\right) dx \\
&= \int_{1/\sqrt{2}}^1 x^2\left[x^2 - 1\right] dx + \int_1^{\sqrt{3/2}} x^2\left(3 - 2x^2\right) dx \\
&= \left[\frac{2x^5}{5} - \frac{x^3}{3}\right]_{1/\sqrt{2}}^1 + \left[x^3 - \frac{2x^5}{5}\right]_1^{\sqrt{3/2}} \\
&= \left(\frac{2}{5} - \frac{1}{3}\right) - \left(\frac{1}{10\sqrt{2}} - \frac{1}{6\sqrt{2}}\right) + \left(\frac{3}{2}\sqrt{\frac{3}{2}} - \frac{18}{20}\sqrt{\frac{3}{2}}\right) \\
&\quad -\left(1 - \frac{2}{5}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{15} + \frac{4}{60\sqrt{2}} + \frac{12}{20}\sqrt{\frac{3}{2}} - \frac{3}{5} \\
&= -\frac{8}{15} + \frac{1}{15\sqrt{2}} + \frac{3}{5}\sqrt{\frac{3}{2}},
\end{aligned}
$$

as before.

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