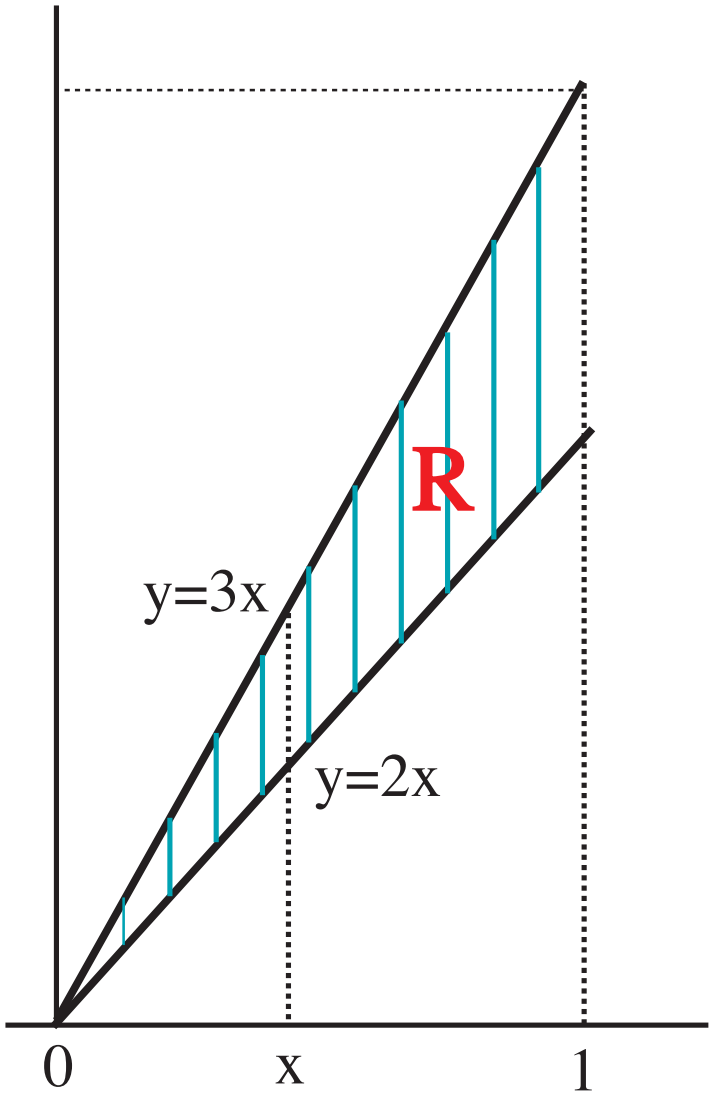
MATH 201: EXAMPLES ON DOUBLE INTEGRALS

April 7, 2009

EXAMPLE. Let

$$R = \{(x, y) : 2x \leq y \leq 3x \text{ and } 0 \leq x \leq 1\}.$$

The Figure indicates the region R of integration.



Let $f : R \longrightarrow \mathbb{R}$ be given by

$$f(x, y) = x^2y + x + 1.$$

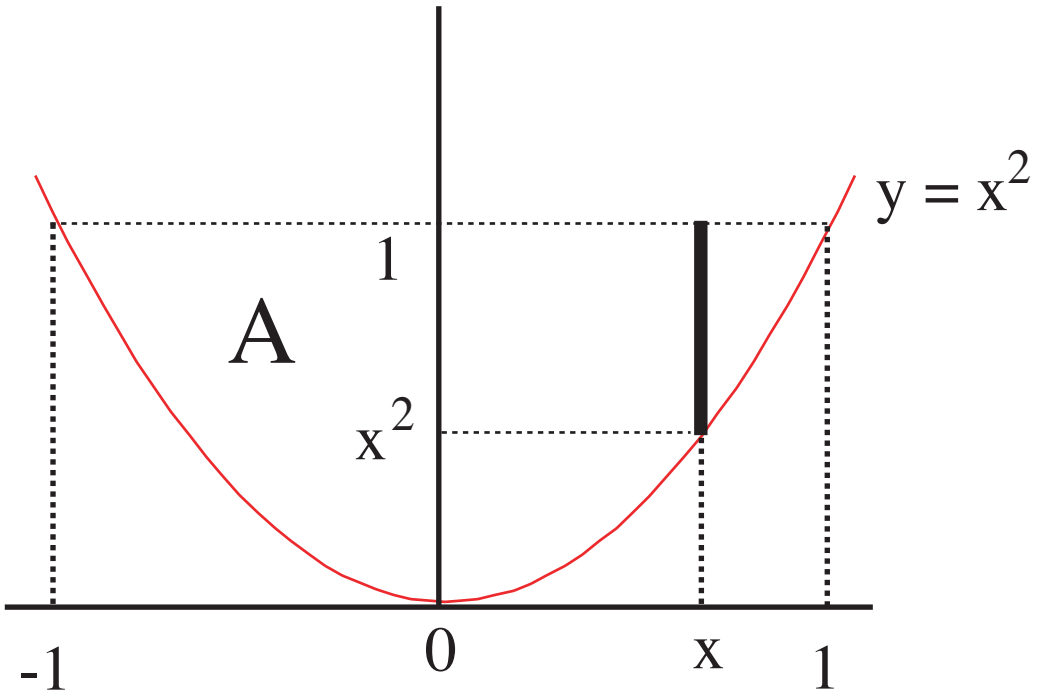
Then we have

$$\begin{aligned}
\int\int_R f &= \int\int_R f(x,y)\ dxdy \\
&= \int_0^1 \left(\int_{2x}^{3x} (x^2 y + x + 1) dy \right) dx \\
&= \int_0^1 \left(\left[x^2 \frac{y^2}{2} + xy + y \right]_{2x}^{3x} \right) dx \\
&= \int_0^1 \left(\frac{9}{2}x^4 + 3x^2 + 3x - 2x^4 - 2x^2 - 2x \right) dx \\
&= \int_0^1 \left(\frac{5}{2}x^4 + x^2 + x \right) dx \\
&= \frac{1}{2} + \frac{1}{3} + \frac{1}{2} \\
&= \frac{8}{6} \\
&= \frac{4}{5}.
\end{aligned}$$

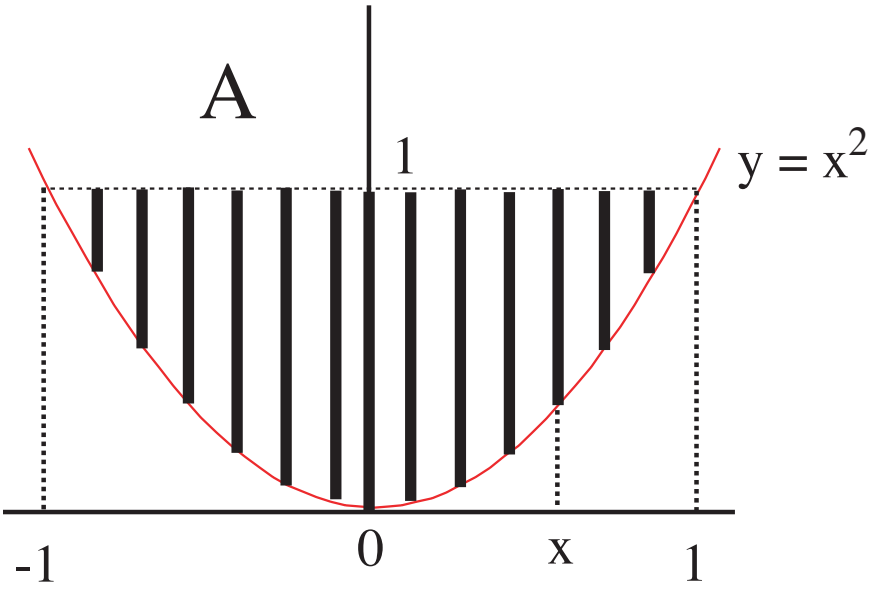
EXAMPLE. Consider the integral

$$\int_{-1}^{1} \left(\int_{x^2}^{1} x^2 e^{y^{5/2}} dy \right) dx.$$

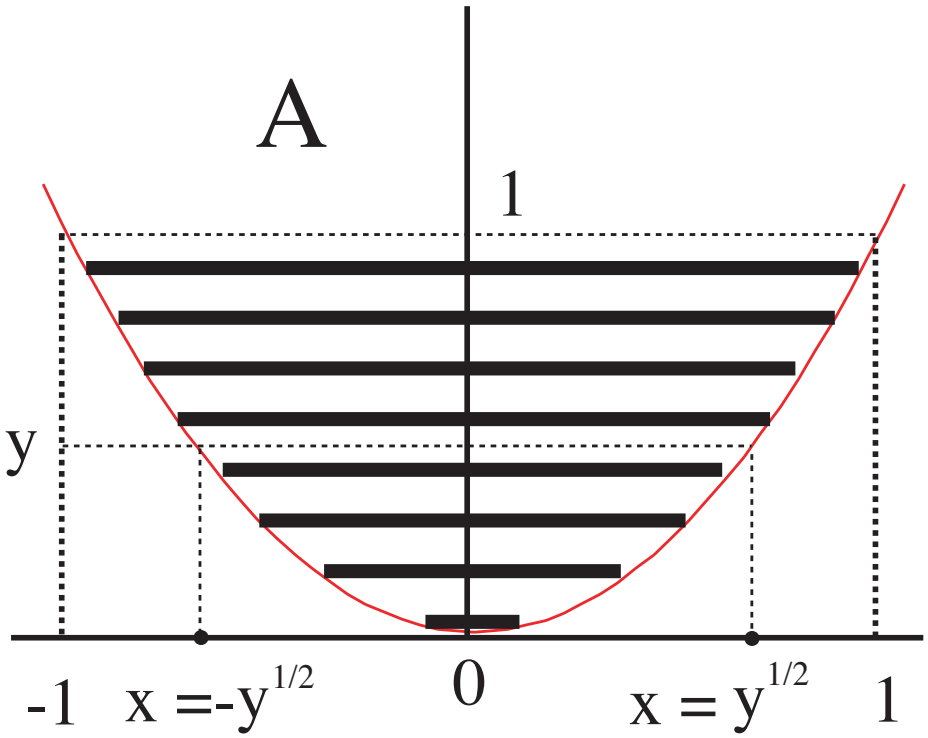
As it stands we cannot evaluate this integral, because we cannot calculate $\int e^{y^{5/2}} dy$. However, we can evaluate it if we interchange the order of integration. First we have to identify the region A of integration.



In the Figure, if x is given between -1 and 1, we see from the integral that the corresponding y values go from x^2 to 1.



In this Figure, as we vary x between -1 and 1, and observe the corresponding y values going from x^2 to 1, we see that the region traced out is the area A in the Figure between the parabola given by $y = x^2$ and $y = 1$.



In the Figure, if y is given between 0 and 1, we see from the preceding Figure that within the area A, the corresponding x values go from $-\sqrt{y}$ to $\sqrt{y}$. Then, as we vary y between 0 and 1, and observe the corresponding x values going from $-\sqrt{y}$ to $\sqrt{y}$, we see that the region traced

out is the same area A between the parabola given by $y = x^2$ and $y = 1$. *This tells us the limits of integration when we change the order of integration.*

We now have, changing the order of integration,

$$\begin{aligned}
\int_{-1}^{1}\left(\int_{x^2}^{1} x^2 e^{y^{5/2}} dy\right) dx &= \int\int_A x^2 e^{y^{5/2}} dx dy \\
&= \int_0^1 \left(\int_{-\sqrt{y}}^{\sqrt{y}} x^2 e^{y^{5/2}} dx\right) dy \\
&= \int_0^1 e^{y^{5/2}} \left[\frac{x^3}{3}\right]_{-\sqrt{y}}^{\sqrt{y}} dy \\
&= \frac{2}{3}\int_0^1 y^{3/2} e^{y^{5/2}} dy \\
&= \frac{2}{3}\int_0^1 \frac{2}{5} e^u du, \quad [u = y^{5/2}] \\
&= \frac{4}{15}[e^u]_0^1 \\
&= \frac{4}{15}(e-1)
\end{aligned}$$

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