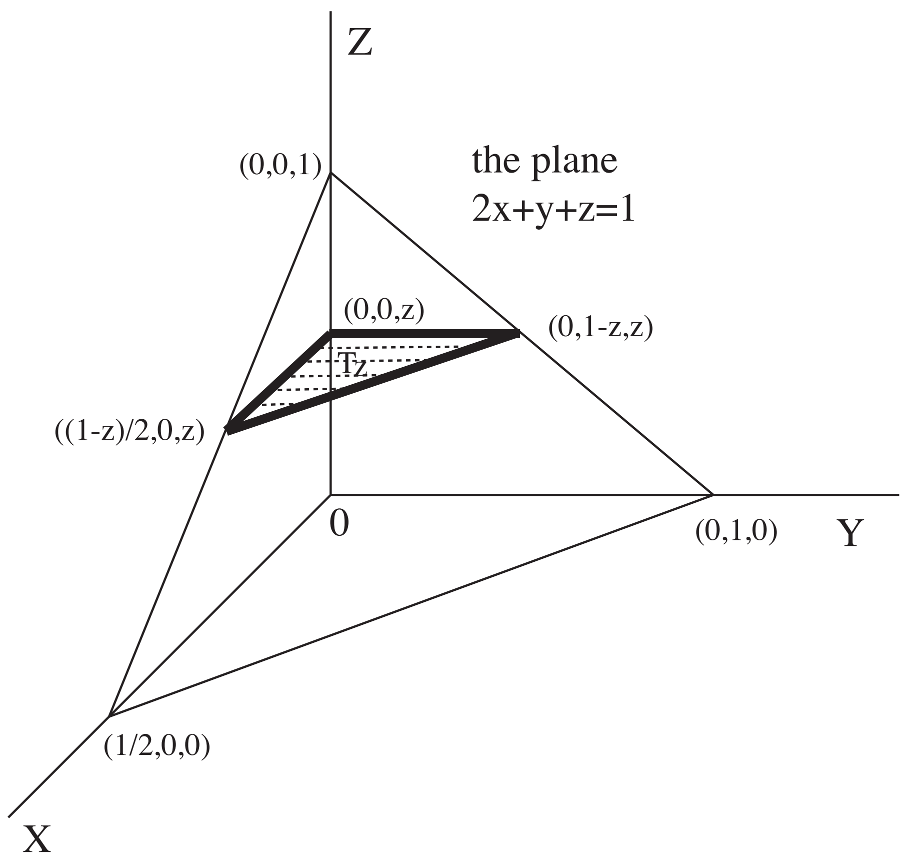
EXAMPLES CONCERNING THE DIVERGENCE THEOREM

EXAMPLE. According to the Divergence Theorem,

$$\int\int_S \langle F, n\rangle dS = \int\int\int_V \mathrm{div} F dxdydz,$$

where V is the 3-dimensional region inside the surface S. Now, consider the surface T consisting of the 4 faces of the tetrahedron with vertices $(0,0,0)$, $(0,0,1)$, $(0,1,0)$ and $(1/2,0,0)$. The upper part of this surface were considered in an earlier example. See the Figure.



We consider the vector field F on T given by

$$F(x,y,z) = (z, x+y, x-y^2+z).$$

Then,

$$\mathrm{div} F(x,y,z) = 0+1+1 = 2.$$

By the Divergence Theorem,

$$\int\int_T \langle F, n\rangle dS = \int\int\int_V 2dxdydz = \int_0^1 \left(\int\int_{A_z} 2dxdy\right) dz,$$

where T_z is the triangular region that is the intersection of the plane whose third coordinate is z with the volume V. Now T_z is a right-angled triangle whose area is

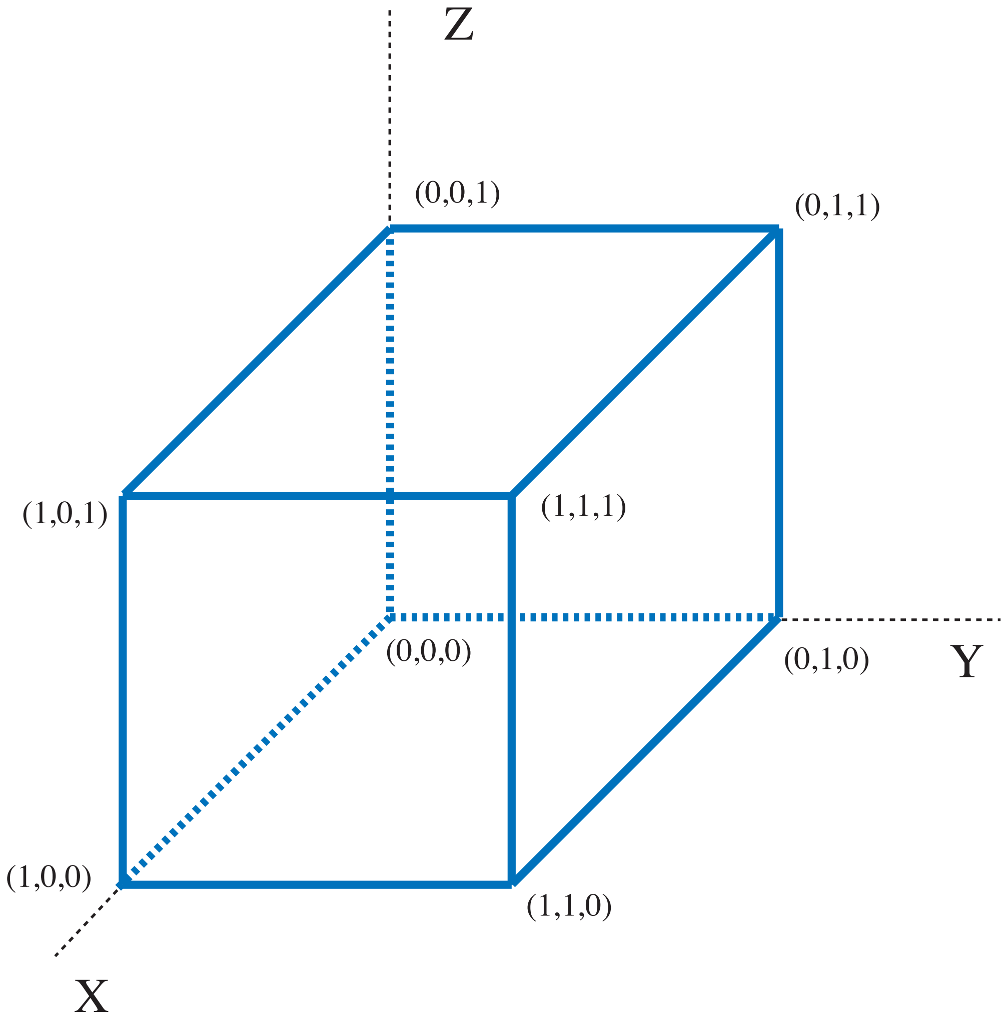
$$\frac{1}{2} \cdot \frac{1-z}{2} \cdot (1-z) = \frac{(1-z)^2}{4}.$$

Thus,

$$\int\int_T \langle F, n \rangle dS = \int_0^1 2 \cdot \frac{(1-z)^2}{4} dz = \int_0^1 \frac{z^2}{2} dz = \frac{1}{6}.$$

EXAMPLE. The Figure depicts the unit cube C which is the solid with 6 flat faces with 8 vertices at $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(1,1,0)$, $(0,1,1)$, $(1,0,1)$ and $(1,1,1)$. The unit cube can be written also as the set

$$[0,1] \times [0,1] \times [0,1] = \Big\{(x,y,z) : 0 \leq x,y,z \leq 1\Big\}.$$



Now, let F be the vector field on $\mathbb{R}^3$ be given by

$$F(x,y,z) = (x,y,z).$$

Thus,

$$\text{div}F(x, y, z) = 1 + 1 + 1 = 3.$$

That is, $F(x, y, z)$ is the position vector of the point (x, y, z). Let S denote the surface of the unit cube. By the Divergence Theorem,

$$\begin{aligned}\int\int_S \langle F, n\rangle dS &= \int\int\int_C \text{div}F(x, y, z)dxdydz \\ &= \int_0^1\int_0^1\int_0^1 \text{div}F(x, y, z)dxdydz \\ &= \int_0^1\left(\int_0^1\left(\int_0^1 3dx\right)dy\right)dz \\ &= 3.\end{aligned}$$

Recall that a vector field F with domain an open set Ω was called *solenoidal* if $\text{div}F = 0$ on Ω. We noted at the time that $\text{curl}F$ is always solenoidal. So, under the conditions of the Divergence Theorem,

$$\begin{aligned}\int\int_S \text{curl}F dS &= \int\int\int_V \text{divcurl}F(x, y, z)dxdydz \\ &= \int\int\int_V 0dxdydz \\ &= 0.\end{aligned}$$

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