

Whiteboard Lecture

Some Basics of Distribution Theory

Conditional Distributions

Let X and Y be two random variables.

1. $f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$.
2. $f_{X,Y}(x,y) = f_{Y|X}(y|x)f_X(x)$.
3. $f_Y(y) = \begin{cases} \sum_x f_{X,Y}(x,y) & X \text{ discrete} \\ \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx & X \text{ continuous} \end{cases}$

Example 1

$$f_{X,Y}(x,y) = x + y, \quad 0 < x < 1, 0 < y < 1.$$

What is $f_{Y|X}(y|x)$?

Answer

$$\begin{aligned} f_X(x) &= \int_0^1 (x + y) dy \\ &= \left[xy + \frac{y^2}{2} \right]_0^1 \\ &= x + \frac{1}{2}, \quad 0 < x < 1. \end{aligned}$$

$$\begin{aligned} \implies f_{Y|X}(y|x) &= \frac{f_{X,Y}(x,y)}{f_X(x)} \\ &= \frac{x + y}{x + \frac{1}{2}} \end{aligned}$$

$$0 < x < 1, \quad 0 < y < 1.$$

Example 2

$$f_{Y|X}(y|x) = \frac{1}{x}, \quad 0 < y < x.$$

$$f_X(x) = \frac{1}{5}, \quad 0 < x < 5.$$

What is f_Y ?

Answer:

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx \\ &= \int_{-\infty}^{\infty} f_{Y|X}(y|x) f_X(x) dx \end{aligned}$$

$$\begin{aligned} \implies f_Y(y) &= \int_y^5 \frac{1}{x} \cdot \frac{1}{5} dx \\ &= \frac{1}{5} \left[\ln(x) \right]_y^5 \\ &= \frac{\ln(5) - \ln(y)}{5}, \quad 0 < y < 5 \end{aligned}$$

Class Exercise

$$\begin{aligned} f_{X,Y}(x,y) &= \frac{6(x^2 + 3y)}{11}, \\ &0 < x < 1, 0 < y < 1. \end{aligned}$$

Determine $f_{Y|X}(y|x)$.

Answer:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy$$

$$\begin{aligned}
&= \int_0^1 \frac{6x^2 + 18y}{11} dy \\
&= \left[\frac{6x^2 y + 9y^2}{11} \right]_0^1 dy \\
&= \frac{6x^2 + 9}{11}, \quad 0 < x < 1.
\end{aligned}$$

i.e. $f_X(x) = \frac{6x^2 + 9}{11}, \quad 0 < x < 1.$

Then

$$\begin{aligned}
f_{Y|X}(y|x) &= \frac{6(x^2 + 3y)}{11} / \frac{6x^2 + 9}{11} \\
&= \frac{2(x^2 + 3y)}{2x^2 + 3} \\
&\text{for } 0 < x < 1, \quad 0 < y < 1.
\end{aligned}$$

Square Bracket Notation

This has become a useful shorthand in 'modern' Statistics literature:

$$f_X(x) = [x]$$

$$f_{X,Y}(x, y) = [x, y]$$

$$f_{Y|X}(y|x) = [y|x]$$

Basic Result 1. is then:

$$1. [y|x] = \frac{[x, y]}{[x]}$$

Example 2 revisited

$$[y|x] = \frac{1}{x}, \quad 0 < y < x.$$

$$[x] = \frac{1}{5}, \quad 0 < x < 5.$$

$$\begin{aligned}
[y] &= \int_{-\infty}^{\infty} [x, y] dx \\
&= \int_{-\infty}^{\infty} [y|x][x] dx
\end{aligned}$$

$$\begin{aligned} &= \int_y^5 \frac{1}{x} \cdot \frac{1}{5} dx \\ &= \frac{\ln(5) - \ln(y)}{5}, \quad 0 < y < 5 \end{aligned}$$

