

## Some Basics of Distribution Theory

### Conditional Distributions

Let  $X$  and  $Y$  be two random variables.

1.  $f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$ .
2.  $f_{X,Y}(x,y) = f_{Y|X}(y|x)f_X(x)$ .
3.  $f_Y(y) = \begin{cases} \sum_x f_{X,Y}(x,y) & X \text{ discrete} \\ \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx & X \text{ continuous} \end{cases}$

### Example 1

$$f_{X,Y}(x,y) = x + y, \quad 0 < x < 1, 0 < y < 1.$$

What is  $f_{Y|X}(y|x)$ ?

### Answer

$$\begin{aligned} f_X(x) &= \int_0^1 (x + y) dy \\ &= \left[ xy + \frac{y^2}{2} \right]_0^1 \\ &= x + \frac{1}{2}, \quad 0 < x < 1. \end{aligned}$$

$$\begin{aligned}\implies f_{Y|X}(y|x) &= \frac{f_{X,Y}(x,y)}{f_X(x)} \\ &= \frac{x+y}{x+\frac{1}{2}}\end{aligned}$$

$$0 < x < 1, 0 < y < 1.$$

### Example 2

$$f_{Y|X}(y|x) = \frac{1}{x}, \quad 0 < y < x.$$

$$f_X(x) = \frac{1}{5}, \quad 0 < x < 5.$$

What is  $f_Y$ ?

Answer:

$$\begin{aligned}f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx \\ &= \int_{-\infty}^{\infty} f_{Y|X}(y|x) f_X(x) dx\end{aligned}$$

$$\begin{aligned}\implies f_Y(y) &= \int_y^5 \frac{1}{x} \cdot \frac{1}{5} dx \\ &= \frac{1}{5} \left[ \ln(x) \right]_y^5 \\ &= \frac{\ln(5) - \ln(y)}{5}, \quad 0 < y < 5\end{aligned}$$

## Class Exercise

$$\text{i.e. } f_X(x) = \frac{6x^2 + 9}{11}, \quad 0 < x < 1.$$

$$f_{X,Y}(x, y) = \frac{6(x^2 + 3y)}{11},$$

$$0 < x < 1, 0 < y < 1.$$

Determine  $f_{Y|X}(y|x)$ .

Answer:

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x, y) dy \\ &= \int_0^1 \frac{6x^2 + 18y}{11} dy \\ &= \left[ \frac{6x^2 y + 9y^2}{11} \right]_0^1 dy \\ &= \frac{6x^2 + 9}{11}, \quad 0 < x < 1. \end{aligned}$$

Then

$$\begin{aligned} f_{Y|X}(y|x) &= \frac{6(x^2 + 3y)}{11} \bigg/ \frac{6x^2 + 9}{11} \\ &= \frac{2(x^2 + 3y)}{2x^2 + 3} \\ &\quad \text{for } 0 < x < 1, \quad 0 < y < 1. \end{aligned}$$

## Square Bracket Notation

This has become a useful shorthand in 'modern' Statistics literature:

$$f_X(x) = [x]$$

$$f_{X,Y}(x, y) = [x, y]$$

$$f_{Y|X}(y|x) = [y|x]$$

Basic Result 1. is then:

$$1. [y|x] = \frac{[x, y]}{[x]}$$

## Example 2 revisited

$$[y|x] = \frac{1}{x}, \quad 0 < y < x.$$

$$[x] = \frac{1}{5}, \quad 0 < x < 5.$$

$$\begin{aligned} [y] &= \int_{-\infty}^{\infty} [x, y] dx \\ &= \int_{-\infty}^{\infty} [y|x][x] dx \\ &= \int_y^5 \frac{1}{x} \cdot \frac{1}{5} dx \\ &= \frac{\ln(5) - \ln(y)}{5}, \quad 0 < y < 5 \end{aligned}$$

