UNIVERSITY OF WOLLONGONG

School of Mathematics and Applied Statistics

STAT902. Advanced Data Analysis

A USEFUL RESULT INVOLVING NORMAL DENSITIES

Definition
Let

$$\phi_{\boldsymbol{\Sigma}}(\boldsymbol{x}) \equiv (2\pi)^{-d/2}|\boldsymbol{\Sigma}|^{-1/2}\exp(-\tfrac{1}{2}\boldsymbol{x}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{x})$$

denote the density of the d-dimensional $N(\mathbf{0}, \boldsymbol{\Sigma})$ distribution.

Result

$$\phi_{\boldsymbol{\Sigma}}(\boldsymbol{x}-\boldsymbol{A}\boldsymbol{a})\phi_{\boldsymbol{\Lambda}}(\boldsymbol{a}) = c\,\phi_{(\boldsymbol{A}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{A}+\boldsymbol{\Lambda}^{-1})^{-1}}\left(\boldsymbol{a}-(\boldsymbol{A}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{A}+\boldsymbol{\Lambda}^{-1})^{-1}\boldsymbol{A}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{x}\right)$$

where the factor c depends only on $\boldsymbol{x}, \boldsymbol{\Sigma}, \boldsymbol{\Lambda}$ and $\boldsymbol{A}$, but not on $\boldsymbol{a}$.

Derivation of Result
We may ignore any multiplicative factors not involving $\boldsymbol{a}$. The left-hand side of the result is proportional to

$$\exp\{-\tfrac{1}{2}(\boldsymbol{x}-\boldsymbol{A}\boldsymbol{a})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{A}\boldsymbol{a})\}\exp(-\tfrac{1}{2}\boldsymbol{a}^T\boldsymbol{\Lambda}^{-1}\boldsymbol{a}) = \exp[-\tfrac{1}{2}\{(\boldsymbol{x}-\boldsymbol{A}\boldsymbol{a})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{A}\boldsymbol{a})+\boldsymbol{a}^T\boldsymbol{\Lambda}^{-1}\boldsymbol{a}\}].$$

Then minus twice the exponent is

$$\begin{aligned}
&(\boldsymbol{x}-\boldsymbol{A}\boldsymbol{a})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{A}\boldsymbol{a})+\boldsymbol{a}^T\boldsymbol{\Lambda}^{-1}\boldsymbol{a}\\
&\quad= \boldsymbol{a}^T(\boldsymbol{A}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{A}+\boldsymbol{\Lambda}^{-1})\boldsymbol{a} - 2\boldsymbol{a}^T\boldsymbol{A}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{x} + \boldsymbol{x}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{x}\\
&\quad= \boldsymbol{a}^T(\boldsymbol{A}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{A}+\boldsymbol{\Lambda}^{-1})\boldsymbol{a} - 2\boldsymbol{a}^T(\boldsymbol{A}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{A}+\boldsymbol{\Lambda}^{-1})(\boldsymbol{A}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{A}+\boldsymbol{\Lambda}^{-1})^{-1}\boldsymbol{A}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{x} + \boldsymbol{x}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{x}\\
&\quad= \boldsymbol{a}^T(\boldsymbol{A}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{A}+\boldsymbol{\Lambda}^{-1})\boldsymbol{a} - 2\boldsymbol{a}^T(\boldsymbol{A}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{A}+\boldsymbol{\Lambda}^{-1})(\boldsymbol{A}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{A}+\boldsymbol{\Lambda}^{-1})^{-1}\boldsymbol{A}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{x}\\
&\qquad\quad+\boldsymbol{x}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{A}(\boldsymbol{A}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{A}+\boldsymbol{\Lambda}^{-1})^{-1}(\boldsymbol{A}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{A}+\boldsymbol{\Lambda}^{-1})(\boldsymbol{A}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{A}+\boldsymbol{\Lambda}^{-1})^{-1}\boldsymbol{A}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{x}\\
&\qquad\quad-\boldsymbol{x}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{A}(\boldsymbol{A}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{A}+\boldsymbol{\Lambda}^{-1})^{-1}\boldsymbol{A}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{x} + \boldsymbol{x}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{x}\\
&\quad= \{\boldsymbol{a}-(\boldsymbol{A}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{A}+\boldsymbol{\Lambda}^{-1})^{-1}\boldsymbol{A}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{x}\}^T(\boldsymbol{A}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{A}+\boldsymbol{\Lambda}^{-1})\{\boldsymbol{a}-(\boldsymbol{A}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{A}+\boldsymbol{\Lambda}^{-1})^{-1}\boldsymbol{A}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{x}\}\\
&\qquad\quad+\text{terms not involving } \boldsymbol{a}
\end{aligned}$$

Therefore $\exp[-\frac{1}{2}\{(\boldsymbol{x}-\boldsymbol{A}\boldsymbol{a})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{A}\boldsymbol{a})+\boldsymbol{a}^T\boldsymbol{\Lambda}^{-1}\boldsymbol{a}\}]$ is proportional to

$$\exp[-\tfrac{1}{2}\{\boldsymbol{a}-(\boldsymbol{A}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{A}+\boldsymbol{\Lambda}^{-1})^{-1}\boldsymbol{A}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{x}\}^T(\boldsymbol{A}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{A}+\boldsymbol{\Lambda}^{-1})\{\boldsymbol{a}-(\boldsymbol{A}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{A}+\boldsymbol{\Lambda}^{-1})^{-1}\boldsymbol{A}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{x}\}]$$

which is proportional to the right-hand side of the result.