UNIVERSITY OF WOLLONGONG

School of Mathematics and Applied Statistics

STAT902. Advanced Data Analysis

SOME RESULTS FROM MULTIVARIATE STATISTICS

Covariance matrices

Let $\mathbf{x}$ be an $n \times 1$ random vector. The *covariance matrix* of $\mathbf{x}$ is an $n \times n$ matrix, denoted $\text{Cov}(\mathbf{x})$, whose (i,j)th entry is the covariance between x_i and x_j. An equivalent definition is

$$\text{Cov}(\mathbf{x}) = \mathsf{E}\Big[\{\mathbf{x} - \mathsf{E}(\mathbf{x})\}\{\mathbf{x} - \mathsf{E}(\mathbf{x})\}^T\Big].$$

If $\mathbf{x}$ is a random vector, $\mathbf{A}$ is a constant matrix and $\mathbf{c}$ is a constant vector whose dimensions are such that $\mathbf{Ax} + \mathbf{c}$ is defined then

$$\mathsf{E}(\mathbf{Ax} + \mathbf{c}) = \mathbf{A}\mathsf{E}(\mathbf{x}) + \mathbf{c}$$

and

$$\text{Cov}(\mathbf{Ax} + \mathbf{c}) = \mathbf{A}\text{Cov}(\mathbf{x})\mathbf{A}^T.$$

The following relationships hold between conditional and unconditional means and covariance matrices:

$$\begin{aligned} \mathsf{E}(\mathbf{y}) &= \mathsf{E}\{\mathsf{E}(\mathbf{y}|\mathbf{x})\} \\ \text{Cov}(\mathbf{y}) &= \mathsf{E}\{\text{Cov}(\mathbf{y}|\mathbf{x})\} + \text{Cov}\{\mathsf{E}(\mathbf{y}|\mathbf{x})\}. \end{aligned}$$

Multivariate normal distribution

The $d \times 1$ random vector $\mathbf{x}$ has a multivariate normal distribution with mean $\boldsymbol{\mu}$ and invertible covariance matrix $\boldsymbol{\Sigma}$ if its probability density function is

$$[\mathbf{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma}] = (2\pi)^{-d/2}|\boldsymbol{\Sigma}|^{-1/2} \exp\{-\tfrac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu})\}$$

and denote this by

$$\mathbf{x} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}).$$

If

$$\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \sim N\left(\begin{bmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{bmatrix}, \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{bmatrix}\right)$$

is a general partitioned normal random vector, then the marginal distribution of $\mathbf{x}_2$ is $N(\mu_2, \boldsymbol{\Sigma}_{22})$ and the conditional distribution of $\mathbf{x}_2$ given $\mathbf{x}_1$ is

$$[\mathbf{x}_2|\mathbf{x}_1] \sim N(\boldsymbol{\mu}_2 + \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}(\mathbf{x}_1 - \boldsymbol{\mu}_1), \boldsymbol{\Sigma}_{22} - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}).$$

Analogous results hold for $\mathbf{x}_1$. Alternatively, $\mathbf{x}_2|\mathbf{x}_1$ may be expressed in terms of the submatrices of the *inverse* covariance matrix of $[\mathbf{x}_1^T, \mathbf{x}_2^T]^T$. If

$$\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \sim N\left(\begin{bmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{bmatrix}, \begin{bmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} \\ \mathbf{Q}_{21} & \mathbf{Q}_{22} \end{bmatrix}^{-1}\right)$$

then

$$[\mathbf{x}_2|\mathbf{x}_1] \sim N(\boldsymbol{\mu}_2 - \mathbf{Q}_{22}^{-1}\mathbf{Q}_{21}(\mathbf{x}_1 - \boldsymbol{\mu}_1), \mathbf{Q}_{22}^{-1}).$$