Advanced Data Analysis

Whiteboard Lecture

Laplace's Method and PQL

Whiteboard Interlude I

Example of Laplace's Method

Consider the integral I given by:

$$I = \int_{-\infty}^{\infty} \frac{100000}{(x^2 - 14x + 58)^5}\, dx$$

Suppose, for the moment, that I can't be solved analytically.

Laplace's Method is a means of approximating I.

First note that

$$I = \int_{-\infty}^{\infty} e^{h(x)}\, dx$$

where

$$h(x) = \ln(100000) - 5\ln(x^2 - 14x + 58).$$

$$\Longrightarrow h'(x) = \frac{-5(2x - 14)}{x^2 - 14x + 58}$$

$$\Longrightarrow h'(x) = 0 \quad \Longleftrightarrow \quad x = 7$$

i.e.

$$x_0 = 7.$$

Next:

$$h''(x) = \frac{(-10)(x^2 - 14x + 58) - (2x - 14)(-5)(2x - 14)}{(x^2 - 14x + 58)^2}$$

Hence

$$h''(x_0) = h''(7) = \frac{(-10)(7^2 - 98 + 58)}{(7^2 - 98 + 58)^2} = \frac{-10}{9}.$$

Laplace's Method says

$$\begin{aligned} I &\simeq e^{h(7)} \sqrt{\frac{2\pi}{-(-10/9)}} \\ &= \left(\frac{10}{9}\right)^5 \sqrt{\frac{9\pi}{5}} \\ &\simeq 4.027 \end{aligned}$$

In fact, I can be solve analytically (e.g. using properties of Student t-distributions).

The exact answer is

$$I = \frac{3500000\,\pi}{2519424} \simeq 4.364.$$

The relative error is

$$\begin{aligned} \frac{|I_{\text{Laplace}} - I|}{I} &= \frac{|4.027 - 4.364|}{4.364} \\ &= 7.7\% \end{aligned}$$

Derivation of Univariate Laplace's Method

$$I = \int_{-\infty}^{\infty} e^{h(x)}\, dx.$$

Taylor's series expansion leads to:

$$\begin{aligned} h(x) &= h(x_0 + x - x_0) \\ &= h(x_0) + (x - x_0)h'(x_0) + \tfrac{1}{2}(x - x_0)^2 h''(x_0) + \ldots \\ &\simeq h(x_0) + (x - x_0)h'(x_0) + \tfrac{1}{2}(x - x_0)^2 h''(x_0) \end{aligned}$$

Choose x_0 to zero out the linear term.

i.e.

$$h'(x_0) = 0.$$

$$\begin{aligned}
I &\simeq \int_{-\infty}^{\infty} e^{h(x_0)+\frac{1}{2}(x-x_0)^2 h''(x_0)}\,dx \\
&= e^{h(x_0)} \int_{-\infty}^{\infty} e^{\frac{1}{2}(x-x_0)^2 h''(x_0)}\,dx \\
&= e^{h(x_0)} \int_{-\infty}^{\infty} e^{\frac{-(x-x_0)^2}{2\left(\sqrt{\frac{1}{-h''(x_0)}}\right)^2}}\,dx \\
&= e^{h(x_0)} \int_{-\infty}^{\infty} e^{\frac{-(x-x_0)^2}{2\left(\sqrt{\frac{1}{-h''(x_0)}}\right)^2}} \frac{1}{\sqrt{2\pi\left(\sqrt{\frac{1}{-h''(x_0)}}\right)^2}}\,dx \\
&\quad \times \sqrt{2\pi\left(\sqrt{\frac{1}{-h''(x_0)}}\right)^2}
\end{aligned}$$

Hence:

$$\begin{aligned}
I &\simeq e^{h(x_0)} \times 1 \times \sqrt{2\pi\left(\sqrt{\frac{1}{-h''(x_0)}}\right)^2} \\
&= e^{h(x_0)} \sqrt{\frac{2\pi}{-h''(x_0)}}
\end{aligned}$$

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