

1 Extremum of a function

The purpose of this section is to quickly *revise* how to find the local maxima or minima of a function of n independent variables. In section 1.1 we consider the easier case of a function of one variable. In section 1.2 we consider the more general case of a function of n variables.

1.1 Extremum in two dimensions

Consider the function $y = f(x)$ shown in figure 1. At which points does the function obtain a local maxima? By inspection they occur at $x = a$ and $x = c$. The former is also the global maximum. Where does a local minima occur? By inspection it occurs at the point $x = b$.

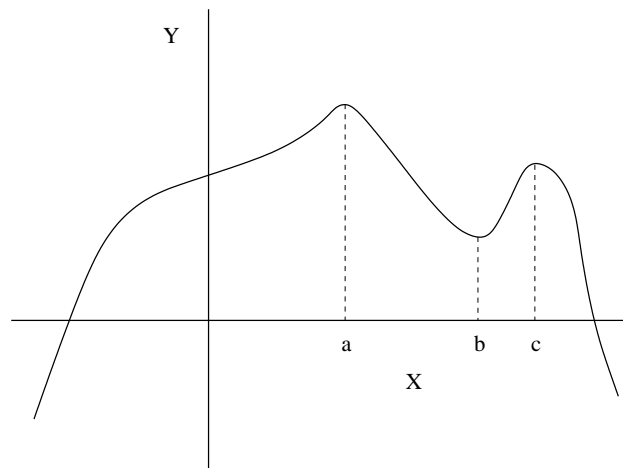


Figure 1: The graph of a function $y = f(x)$.

If instead of a graph you were given a function $y = f(x)$ how would you find the local extrema? These points occur when

Local Maxima

$$\begin{aligned} \frac{dy}{dx} &= 0 \\ \frac{d^2y}{dx^2} &> 0 \end{aligned}$$

Local Minima

$$\begin{aligned} \frac{dy}{dx} &= 0 \\ \frac{d^2y}{dx^2} &< 0. \end{aligned}$$

Suppose that the function is

$$y = x^3/3 - 2x^2 + 3.$$

A little calculation shows that there is a local maxima at the point $x = 0$ and a local minima at the point $x = 4$. The graph of this function is shown in figure 2.

Consider a series of slightly different questions.

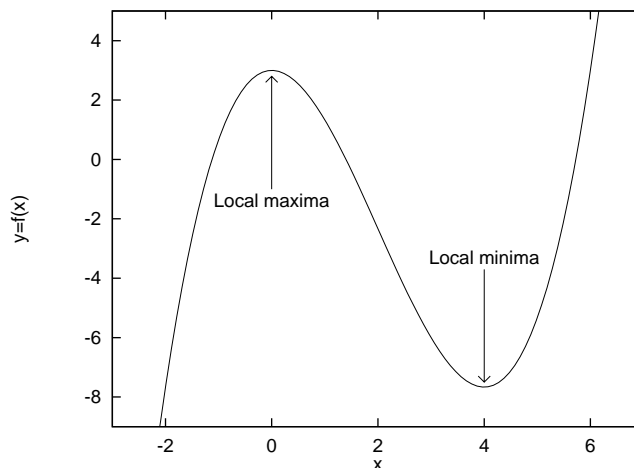


Figure 2: The graph of the function $y = x^3/3 - 2x^2 + 3$.

- Q.** What is the maximum value of the function $y = x^3/3 - 2x^2 + 3$ between the points $x = -1$ and $x = 1$?
- A.** From the graph we see that the maximum value occurs at the local maxima $(x = 0, y = 3)$.
- Q.** What is the maximum value between the points $x = -1$ and $x = 7$?
- A.** Although the the *local maxima* at $x = 0$ is in the range $-1 \leq x \leq 7$ the function takes its maximum value at one of the end points: $(x = 7, y = 58/3)$.
- Q.** What is the minimum value of the function $y = x^3/3 - 2x^2 + 3$ between the points $x = 1$ and $x = 2$?
- A.** There are no local minima in this range, the minimum value occurs at the end point $(x = 2, y = -7/3)$.
- Q.** What is the minimum value between the points $x = 1$ and $x = 5$?
- A.** A local minima occurs at the point $x = 4$. At this point the function is at its minimum value $(y = -23/3)$ over the range $1 \leq x \leq 5$.
- Q.** What is the minimum value between the points $x = -2$ and $x = 5$?
- A.** Although the *local minima* at $x = 4$ is in the range $-2 \leq x \leq 5$ the function takes its minimum value at one of the end points: $(x = -2, y = -23/3)$.

More generally, if were are given a function $y = g(x)$ how do we find the maximum value of y between the points $x = a$ and $x = b$?

1. If there are *no local maxima* in the range $a \leq x \leq b$ then the maximum value of the function must either occur at $x = a$ or $x = b$ (or possible both). Calculate $y = g(a)$ and $y = g(b)$ and determine which is the larger.

2. Suppose that there are, for example, two local maxima in the range $a \leq x \leq b$. Suppose that they are located at $x = \alpha$ and $x = \beta$. Then the maximum of the function $y = g(x)$ will occur either at one of the local maxima (or possible both simultaneously) or at one of the ends points (or possible both simultaneously). (It could even happen that $g(a) - g(b) = g(\alpha)g(\beta)$). Thus to find the maximum of the function calculate $y = g(a)$, $y = g(\alpha)$, $y = g(\beta)$ and $y = g(b)$ and determine which is the larger.

1.2 Extrema in more than two dimensions

In section 1.1 we considered the case of a function that depended upon one variable. In this section we consider the more general case where the function depends upon n variables. This means that instead of plotting a graph in two dimensions we have to plot one in $n + 1$ dimensions — not very easy!

Let $\mathcal{F} = f(x) = f(x_1, x_2, \dots, x_n)$ be a function of n variables $x = (x_1, \dots, x_n)$. If the n variables are independent, then local extrema of f (a minimum, maximum or saddle point) can be found by finding where the gradient of f is zero, i.e. by solving the n equations

$$\frac{\partial \mathcal{F}}{\partial x_1} = \dots = \frac{\partial \mathcal{F}}{\partial x_n} = 0.$$

How do we distinguish between a maxima and a minima? To answer this question we limit ourselves to functions of *two* variables.

Theorem 1 *Suppose that at the point $(x, y) = (a, b)$ the function $\mathcal{F} = f(x, y)$ has the following properties*

$$\frac{\partial \mathcal{F}}{\partial x} = \frac{\partial \mathcal{F}}{\partial y} = 0 \quad \text{and} \quad \left(\frac{\partial^2 \mathcal{F}}{\partial x y} \right)^2 - \frac{\partial^2 \mathcal{F}}{\partial x^2} \frac{\partial^2 \mathcal{F}}{\partial y^2} \equiv \mathcal{D},$$

Then if

1. $\mathcal{F}_{xx}(a, b) < 0$ and $\mathcal{D} < 0$, then \mathcal{F} has a local maxima at the point (a, b) .
2. $\mathcal{F}_{xx}(a, b) > 0$ and $\mathcal{D} < 0$, then \mathcal{F} has a local minima at the point (a, b) .
3. $\mathcal{D} > 0$, then \mathcal{F} has neither a local maxima nor a local minima at the point (a, b) .

Suppose that we want the extrema of a function of two variables in the region $x^2 + y^2 \leq 4$. These points either occur in the interior of the region ($x^2 + y^2 < 4$) or on its boundary ($x^2 + y^2 = 4$). We can find out if any extrema occur in the interior by applying Theorem 1. That leaves us with the following problem

$$\text{Maximise the function} \quad \mathcal{F}(x, y) \quad \text{subject to the constraint} \quad x^2 + y^2 = 4.$$

More generally we have the problem,

$$\text{Maximise the function} \quad \mathcal{F}(x, y) \quad \text{subject to the constraint} \quad g(x, y) = 0.$$

In order to solve this problem we will use the technique of *Lagrange Multipliers*. Once we have found the maximum value on the boundary we can answer our question by comparing its value against that of any local maxima in the interior (should they exist).

2 Lagrange Multipliers

Motivated by the discussion at the end of the previous section we would like to find the local maxima or minima of a function $f(x, y)$ subject to a constraint. The constraints restrict the inputs (x and y) to lie on an implicitly-defined surface $g(x, y) = k$. Such problems are solved by using the method of Lagrange¹ multipliers.

What kinds of real problems can be solved using Lagrange multipliers?

Aeronautical Engineering Suppose that you are designing an aircraft wing. By changing its shape you change the drag on it. Your problem is to minimise the drag but there is a constraint on what the force on the airfoil has to be.

Chemical Engineering/Economics Lagrange multipliers are used in the large industrial optimization systems that are implemented in many refineries today. The aim is to run the process units in a way that makes more profit.

Chemical Engineering/Physical Chemistry Calculation of chemical equilibria by minimising Gibbs free energy subject to the stoichiometric constraints

Mechanical Engineering

1. Contact problems, where the motion of one body is constrained with respect to another i.e. the bodies cannot impinge upon each other. Physically the multiplier represents the constraint force necessary to restrain motion.
2. Any problem in structural mechanics where motion is constrained e.g. gears, sliding motion, slots, etc. can be modelled using Lagrange multipliers. The constraint usually represents a kinematic condition, such as a joint, between two parts in a mechanism, e.g. a universal joint in a vehicle's drivetrain. In this example the Lagrange multiplier represents the force (or torque) supported by the joint. These forces can be used to determine the stresses in a part of a mechanism. Consequently you can predict the dynamic conditions under which the part will fail (break).
3. Mechanics of deformable bodies.

Economics This technique is pervasive in economics, where many problems relate to the best use of scarce resources.

1. In demand theory, the consumer is assumed to maximise utility subject to prevailing prices and income by choosing the right combination of goods to buy.

¹Joseph Louis Lagrange (1736-1813) was a French mathematician.

2. Firms are assumed to maximise profits subject to the constraints of technology and the production function by setting their output and price at the right levels.

The power of the method lies in the strictly mechanical nature of the process:

- you compute something,
- plug in something, and
- Chug out the answer.

The method of Lagrange Multipliers is explained in section 2.2 and applied to some examples in section 2.3. Before proceeding to the method I will first motivate the mathematics behind it in section 2.1.

2.1 Lagrange Multipliers: Motivation of the method

2.1.1 Example one

Let's consider the example of trying to maximise the production of a firm under a budget constraint. Suppose the firm's production, f , is a function of two variables, x and y , representing labour costs and capital costs, and is given by

$$f(x, y) = x^{2/3}y^{1/3}.$$

Suppose that the labour costs are p_1 per unit labour and the capital costs are p_2 per unit capital.

What is the maximum output that can be achieved with a budget of c dollars?

If we want to maximise f without regard to the budget, we simply increase x and y as far as we can. Furthermore, given the choice we would choose to increase x rather than y . However, the budget will prevent us from increasing x and y beyond a certain point. With prices of p_1 and p_2 , the amount spent on x is p_1x and the amount spent on y is p_2y , so we must have

$$g(x, y) = p_1x + p_2y \leq c,$$

where $g(x, y)$ is the total cost of the inputs x and y . Let's consider the case when $p_1 = p_2 = 100$ and $c = 378$. Then

$$100x + 100y \leq 378$$

so

$$x + y \leq 3.78.$$

Suppose that we decide to spend all the budget. Then the budget constraint is

$$x + y = 3.78.$$

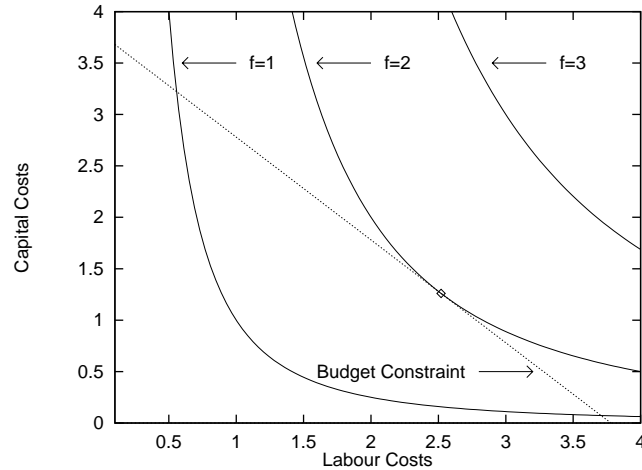


Figure 3: Level curves of production output ($f(x, y) = k$) and budget constraint ($g(x, y)$).

So our problem is to maximise the output of the firm (maximise the function $f(x, y)$) without overspending the budget (subject to the constraint $g(x, y)$).

Figure 3 shows the constraint function g and some contours of f . A point of intersection of the curve g with the contour f is a possible solution of our problem: the intersection represents an output satisfying the constraint. Our aim is to maximise f , so we want to find the point which lies on the level curve with the largest possible f value *and* which lies within the budget. The point we are looking for must lie on the budget line.

Unless we are at the point where the budget constraint is *tangent* to the contour $f = 2$, we can always increase f by moving along the line representing the budget constraint. For example, if we are to the left of the point of tangency, moving right will increase f ; if we are to the right of the point of tangency, moving left will increase f . Thus, the maximum value of f on the budget constraint occurs at the point where the budget constraint is tangent to the contour of f .

Suppose that there is a point $\mathcal{P} = (x, y)$ where the output function is tangent to the budget line. Then at this point the gradient of f , given by $\text{grad } f$, and the normal to the budget line, given by $\text{grad } g$, must be parallel *parallel*. Thus there is a scalar λ , called the *Lagrange multiplier*, so that

$$\text{grad } f = \lambda \text{ grad } g$$

2.1.2 Example two

Suppose we wish to optimize an *objective function* $f(x, y)$ subject to a *constraint* $g(x, y) = c$. Then assuming f is defined at the point (x_0, y_0) we say that

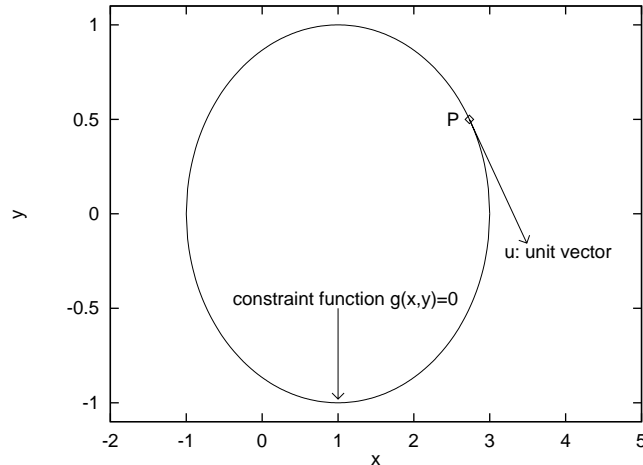


Figure 4: The location of a hypothetical local maximum at the point P on the constraint curve $g(x, y) = 0$.

- The function f has a **local maximum** at the point (x_0, y_0) , **subject to the constraint** $g(x, y) = c$, if $f(x, y) \leq f(x_0, y_0)$ for all (x, y) near (x_0, y_0) such that $g(x, y) = c$.
- The function f has a **local minimum** at the point (x_0, y_0) , **subject to the constraint** $g(x, y) = c$, if $f(x, y) \geq f(x_0, y_0)$ for all (x, y) near (x_0, y_0) such that $g(x, y) = c$.
- The function f has a **global maximum** at the point (x_0, y_0) , **subject to the constraint** $g(x, y) = c$, if $f(x, y) \leq f(x_0, y_0)$ for all (x, y) such that $g(x, y) = c$.
- The function f has a **global minimum** at the point (x_0, y_0) , **subject to the constraint** $g(x, y) = c$, if $f(x, y) \geq f(x_0, y_0)$ for all (x, y) such that $g(x, y) = c$.

Suppose that we know that the function f has a local maximum on the constraint curve $g(x, y) = 0$ at a point $P = (x_0, y_0)$. (The argument in the case of a local minimum is the same). Let u be a unit vector which is tangent to $g(x, y)$ at the point P . This situation is shown in figure 4. What can we deduce about the function f at the point P ?

Let u be a unit vector which is tangent on the curve $g(x, y) = 0$ at the point P . Recall that the *directional derivative* of a function along a vector v gives the rate of change of f in that direction. We want to know how the function f changes along the curve $g(x, y) = 0$ at the point P . Thus the appropriate vector v to use is the tangent vector at the point P (u).

The directional derivative of the function f along the tangent direction at the point P is given by

$$f_u(x_0, y_0) = \text{grad } f(x_0, y_0) \cdot u$$

If this number were positive then by moving along the curve $g(x, y) = 0$ in the direction of u we would increase f . This can not be the case because the point P is a local maximum. Thus $f_u(x_0, y_0) \leq 0$. If the directional derivative were negative, we could increase f by moving along $g(x, y) = 0$ in the direction $-u$. This can not be the case because the point P is a local maximum. Thus $f_u(x_0, y_0) \geq 0$. Thus, if the function $f(x, y)$ has a local

maximum on the constraint curve $g(x, y) = 0$ at the point $P = (x_0, y_0)$ we must have have

$$\begin{aligned} f_u(x_0, y_0) &= 0, \\ \Rightarrow \text{grad } f(x_0, y_0) \cdot u &= 0 \end{aligned}$$

Remember that the directional derivative of a function f in the direction of a unit vector u can also be written in the form

$$\begin{aligned} f_u(a, b) &= \text{grad } f(a, b) \cdot u, \\ &= \|\text{grad } f(a, b)\| \cos \theta, \end{aligned}$$

where θ is the angle between the unit vector and the gradient vector. Thus assuming that $\text{grad } f \neq \mathbf{0}$ then the directional derivative is zero whenever the unit vector is perpendicular to the gradient vector. Thus we have shown that at a local maximum the gradient vector is perpendicular to the tangent vector.

Now consider the directional derivative of the constraint $g(x, y) = 0$ at the point P in the direction u . The function g takes the same value along the constraint curve ($g = 0$ by definition!). Therefore the directional derivative of g along the constraint curve must be zero (else it would not be constant!). Thus

$$\text{grad } g(x_0, y_0) \cdot u = 0.$$

We have therefore shown that the vectors $\text{grad } f$ and $\text{grad } g$ are both perpendicular to the unit tangent vector u . Consequently these vectors must be parallel. Thus, if (x_0, y_0) is a local maximum or minimum, subject to the constraint $g(x, y) = c$, and provided that $\text{grad } g(x_0, y_0)$ is non-zero, there is a scalar λ such that

$$\text{grad } f(x_0, y_0) = \lambda \text{grad } g(x_0, y_0).$$

In summary, we have:

If $f(x, y)$ has a local maximum or minimum subject to the constraint $g(x, y) = 0$ at a point (x_0, y_0) and $\text{grad } g(x_0, y_0)$ is non-zero, then there is a scalar λ such that

$$\boxed{\text{grad } f(x_0, y_0) = \lambda \text{grad } g(x_0, y_0).}$$

2.2 Lagrange multipliers: The method

To solve a problem with Lagrange multipliers, use the following procedure. For the sake of clarity, I consider a problem with three variables: x , y , and z .

1. Write down the quantity to be maximised or minimised. This function $f(x, y, z)$ is known as the **objective function**.
2. Write down the **constraint** $g(x, y, z) = 0$.

3. Form a function to differentiate:

$$\begin{aligned}\mathcal{F}(x, y, z, \lambda) &= (\text{objective function}) - \lambda (\text{constraint}), \\ &= f(x, y, z) - \lambda g(x, y, z).\end{aligned}$$

4. Find the partials of \mathcal{F} with respect to x , y , z , and λ . Set these equal to zero.

5. Solve the system of equations that results. The values of the variables from the solution provide the desired maximum or minimum.

6. If the problem has only one solution draw a graph of the constraint g and some level curves of the objective function f . It should then be clear if the point is a maxima or a minima ². If there is more than one solution, compute the value of the objective function at each solution point and take the point(s) that gives the maximum or minimum.

Suppose that we have a function of two variables $f(x, y)$ and a constraint equation $g(x, y) = 0$. Then the extrema point(s) satisfy the following equations.

$$\mathcal{F}_x = f_x - \lambda g_x = 0, \quad (1)$$

$$\mathcal{F}_y = f_y - \lambda g_y = 0, \quad (2)$$

$$\mathcal{F}_\lambda = g(x, y) = 0. \quad (3)$$

What do these equations mean? Equations (1 & 2) are just the conditions for the gradient vector of f and the gradient vector of g to be parallel. We have already seen that this condition is satisfied at an extrema. Equation 3 is simply the equation for the constraint — the solution point must be on the constraint curve.

This method can be generalised, and is particularly useful, to cases when more than one constraint is involved.

2.3 Lagrange multipliers: Some examples

2.3.1 Example one

Let us return to the problem of a firm trying to maximise its output

$$f(x, y) = x^{2/3}y^{1/3},$$

subject to the constraint

$$x + y = 3.78.$$

Apply the method given in section 2.2

²There is a second derivative test for classifying the critical points of constrained optimization problems, but it is rather complicated and not needed in this course.

1. **Objective function:** $f(x, y) = x^{2/3}y^{1/3}$

2. **Constraint function**

$$\begin{aligned} 3.78 &= x + y, \\ g(x, y) = 0 &= x + 2y - 3.78 \end{aligned}$$

3. Form a function to differentiate: $\mathcal{F} = x^{2/3}y^{1/3} - \lambda(x + y - 3.78)$.

4. Find the partials of \mathcal{F} with respect to x , y , z , and λ . Set these equal to zero.

$$\begin{aligned} \mathcal{F}_x = 0 &= \frac{2}{3}x^{-1/3}y^{1/3} - \lambda, \\ \mathcal{F}_y = 0 &= \frac{1}{3}x^{2/3}y^{-2/3} - \lambda, \\ \mathcal{F}_\lambda = 0 &= x + y - 3.78, \end{aligned}$$

5. Solve the system of equations that results. The values of the variables from the solution provide the desired maximum or minimum.

$$\begin{aligned} \mathcal{F}_x = 0 &\Rightarrow \lambda = \frac{2}{3}x^{-1/3}y^{1/3}, \\ \mathcal{F}_y = 0 &\Rightarrow \lambda = \frac{1}{3}x^{2/3}y^{-2/3}, \end{aligned}$$

Solving these expressions for λ gives

$$\begin{aligned} \frac{2}{3}x^{-1/3}y^{1/3} &= \frac{1}{3}x^{2/3}y^{-2/3}, \\ \Rightarrow x &= 2y. \end{aligned}$$

Substituting this value into the equations for \mathcal{F}_λ gives

$$\begin{aligned} 2y + y &= 3.78, \\ \Rightarrow y &= 1.26. \end{aligned}$$

The value of x is given by $x = 2y$ and is therefore 2.52. The maximum output is therefore

$$\begin{aligned} f(2.52, 1.26) &= (2.52)^{2/3}(1.26)^{1/3}, \\ &= 2. \end{aligned}$$

As there is only one answer we should draw a graph of the constraint curve ($g = 0$) and some level curves of the objective function (f). From figure 3 it is clear that this point is a maximum, agreeing with our earlier graphical argument.

2.3.2 Example Two

Maximise the function

$$z = xy,$$

subject to the constraint

$$x + y - 10 = 0.$$

Apply the method given in section 2.2.

1. **Objective function:** xy .
2. **Constraint function** $g(x, y) = x + y - 10 = 0$.
3. Form a function to differentiate: $\mathcal{F} = xy - \lambda(x + y - 10)$.
4. Find the partials of \mathcal{F} with respect to x , y , z , and λ . Set these equal to zero.

$$\mathcal{F}_x = 0 = y - \lambda,$$

$$\mathcal{F}_y = 0 = x - \lambda,$$

$$\mathcal{F}_\lambda = 0 = x + y - 10,$$

5. Solve the system of equations that results. The values of the variables from the solution provide the desired maximum or minimum.

From the equations $\mathcal{F}_x = 0$ and $\mathcal{F}_y = 0$ we have the expressions $y = \lambda$ and $x = \lambda$. Substituting these into the equation for \mathcal{F}_λ gives

$$\begin{aligned} 0 &= 2\lambda - 10, \\ \Rightarrow \lambda &= 5. \end{aligned}$$

From the equation for $\mathcal{F}_x = 0$ we therefore have $y = 5$. From the equation for $\mathcal{F}_y = 0$ we therefore have $x = 5$.

Thus the maximum value occurs at the point $(x, y, \lambda) = (5, 5, 5)$ and is given by $z = 25$.

The solution to this problem is illustrated graphically in figure 5. Note that at the solution the objective function $f(x, y)$ is tangent to the constraint curve $g(x, y)$, i.e. $\text{grad } f$ and $\text{grad } g$ are parallel

2.3.3 Example Three

Find the maximum and minimum values of the function

$$f(x, y) = x + y,$$

on the circle

$$x^2 + y^2 = 4.$$

Apply the method given in section 2.2.

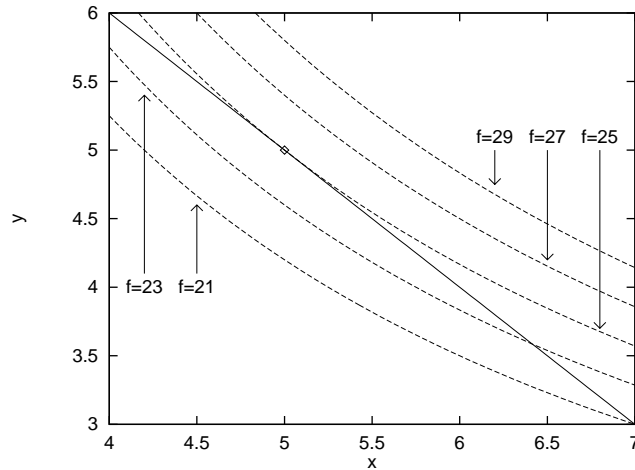


Figure 5: Level surface of the function $f(x, y) = xy$ showing its maximum value on the constraint surface $g(x, y) = x + y - 10$.

1. **Objective function:** $f(x, y) = x + y$

2. **Constraint function**

$$\begin{aligned}x^2 + y^2 &= 4, \\g(x, y) = 0 &= x^2 + y^2 - 4\end{aligned}$$

3. Form a function to differentiate: $\mathcal{F} = x + y - \lambda(x^2 + y^2 - 4)$.

4. Find the partials of \mathcal{F} with respect to x , y , z , and λ . Set these equal to zero.

$$\begin{aligned}\mathcal{F}_x = 0 &= 1 - 2\lambda x, \\ \mathcal{F}_y = 0 &= 1 - 2\lambda y, \\ \mathcal{F}_\lambda = 0 &= x^2 + y^2 - 4,\end{aligned}$$

5. Solve the system of equations that results. The values of the variables from the solution provide the desired maximum or minimum.

$$\begin{aligned}\mathcal{F}_x = 0 &\Rightarrow x = \frac{1}{2\lambda}, \\ \mathcal{F}_y = 0 &\Rightarrow y = \frac{1}{2\lambda},\end{aligned}$$

Substituting these expressions into the equation for \mathcal{F}_λ gives

$$\begin{aligned}4 &= \frac{1}{4\lambda^2} + \frac{1}{4\lambda^2}, \\ \Rightarrow \lambda &= \pm \frac{1}{2\sqrt{2}}.\end{aligned}$$

Thus there are two solution pairs

$$\begin{aligned}(x, y, \lambda) &= \left(\sqrt{2}, \sqrt{2}, \frac{1}{2\sqrt{2}}\right), \\ (x, y, \lambda) &= \left(-\sqrt{2}, -\sqrt{2}, -\frac{1}{2\sqrt{2}}\right).\end{aligned}$$

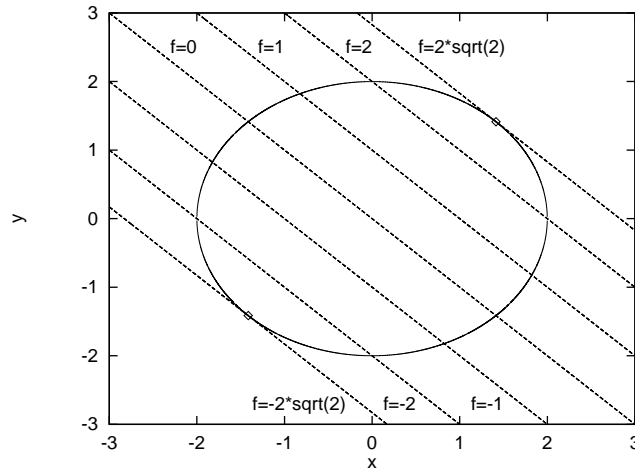


Figure 6: Level surfaces of the function $f(x, y) = x + y$ showing the location of its maximum and minimum values on the constraint surface $x^2 + y^2 = 4$.

At the first solution $f\left(\sqrt{2}, \sqrt{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}\right) = 2\sqrt{2}$. At the second solution $f\left(-\sqrt{2}, -\sqrt{2}, -\frac{1}{2}, \frac{1}{\sqrt{2}}\right) = -2\sqrt{2}$.

Thus the first solution is a maximum whilst the second is a minimum.

The solution to this problem is illustrated graphically in figure 6. Note that at the two extremum of the function the objective function $f(x, y)$ is tangent to the constraint curve $g(x, y)$, i.e. $\text{grad } f$ and $\text{grad } g$ are parallel.

3 References

Links to some useful web pages are provided on my web pages. These include sites showing how to use Maple to gain an insight into the geometry of the method. I also found the following book useful.

W.G. McCallum, D. Hughes-Hallett, A.M. Gleason *et al.* *Multivariable Calculus*, preliminary edition. John Wiley & Sons, 1996.

4 Examples

Examples are taken from the web page

<http://www.math.ucdavis.edu/~hom/calculus/Lagrange/lagrange.html#1>

1. Find the dimensions of the largest rectangle of perimeter p .
2. Find the points on the surface $z = xy + 5$ that are closest to the origin.
3. Find the points on the surface $xy - z^2 = 1$ that are closest to the origin.
4. The temperature at a point (x, y) on a metal plate is give by $4x^2 - 4xy + y^2$. An ant, walking on the plate, traverses a circle of radius 6 centered at the origin. What are the highest and lowest temperatures encountered by the ant?
5. Maximize $x + 2y + 3z$ subject to the constraints $x^2 + y^2 + z^2 = 1$ and $x + y + z = 1$.
6. Find the dimensions of the rectangular box, open at the top, whose volume is 32 cubic feet and which requires the least material for its construction.

5 Solutions

1. Find the dimensions of the largest rectangle of perimeter p .

1. **Objective function:** $f(x, y) = xy$

2. **Constraint function**

$$\begin{aligned} P &= 2x + 2y, \\ g(x, y) &= 0 = 2x - 2y - P \end{aligned}$$

3. Form a function to differentiate: $\mathcal{F} = xy - \lambda(2x - 2y - P)$.

4. Find the partials of \mathcal{F} with respect to x , y , z , and λ . Set these equal to zero.

$$\begin{aligned} \mathcal{F}_x &= 0 = y - 2\lambda, \\ \mathcal{F}_y &= 0 = x - 2\lambda, \\ \mathcal{F}_\lambda &= 0 = 2x + 2y - 8, \end{aligned}$$

5. Solve the system of equations that results. The values of the variables from the solution provide the desired maximum or minimum.

$$\begin{aligned} \mathcal{F}_x = 0 &\Rightarrow y = 2\lambda, \\ \mathcal{F}_y = 0 &\Rightarrow x = 2\lambda, \end{aligned}$$

Substituting these expressions into the equation for \mathcal{F}_λ gives

$$\begin{aligned} 0 &= 4\lambda + 4\lambda - 8, \\ \lambda &= \frac{8}{8} \\ \Rightarrow y &= \frac{8}{4}, \\ x &= \frac{8}{4} \end{aligned}$$

2. Find the points on the surface $\mathbf{z} = \mathbf{xy} + 5$ that are closest to the origin.

1. **Objective function:** $x^2 + y^2 + z^2$.

2. **Constraint function**

$$\begin{aligned} z &= xy + 5, \\ g(x, y) &= 0 = z - xy - 5 \end{aligned}$$

3. Form a function to differentiate: $\mathcal{F} = x^2 + y^2 + z^2 - \lambda(z - xy - 5)$.

4. Find the partials of \mathcal{F} with respect to x , y , z , and λ . Set these equal to zero.

$$\begin{aligned}\mathcal{F}_x &= 0 = 2x + \lambda y, \\ \mathcal{F}_y &= 0 = 2y + \lambda x, \\ \mathcal{F}_z &= 0 = 2z - \lambda, \\ \mathcal{F}_\lambda &= 0 = -(z - xy - 5),\end{aligned}$$

5. Solve the system of equations that results. The values of the variables from the solution provide the desired maximum or minimum.

From the equation $\mathcal{F}_z = 0$ we have the expression $2z = \lambda$. Substituting this expression into the other partials gives

$$\begin{aligned}\mathcal{F}_x = 0 &\Rightarrow x + zy = 0, \\ \mathcal{F}_y = 0 &\Rightarrow y + zx = 0, \\ \mathcal{F}_\lambda = 0 &\Rightarrow z - xy - 5 = 0.\end{aligned}$$

From the equation for $\mathcal{F}_x = 0$ note that $z = 0 \Rightarrow x = 0$. From the equation for $\mathcal{F}_y = 0$ note that $z = 0 \Rightarrow y = 0$. However the point $(x, y, z) = (0, 0, 0)$ does not satisfy the equation for $\mathcal{F}_\lambda = 0$. Thus $z = 0$ is not a solution.

From the equation $\mathcal{F}_x = 0$ we have the expression $y = -\frac{x}{z}$. (We have just shown that $z \neq 0$). Substituting this expression into the remaining partials gives

$$\begin{aligned}\mathcal{F}_y = 0 &\Rightarrow x(z^2 - 1), \\ \mathcal{F}_\lambda = 0 &\Rightarrow z(z - 5) + x^2.\end{aligned}$$

The equation for \mathcal{F}_y says that either $x = 0$ or $z = \pm 1$. Thus there are three cases to consider.

Case 1: $x = 0$.

The equation for \mathcal{F}_λ now implies that either $z = 0$ or $z = 5$. However, we have already shown that $z \neq 0$. Thus a possible maximum/minimum is given by the solution $(x, y, z) = (0, y, 5)$. Recall that $y = -\frac{x}{z}$. Thus the solution is

$$(x, y, z) = (0, 0, 5).$$

Case 2: $z = 1$.

The equation for \mathcal{F}_λ now implies that $x = \pm 2$. When $x = 2$ we have $y = -2$. When $x = -2$ we have $y = 2$. Thus we have the solution pairs

$$(x, y, z) = (2, -2, 1) \text{ and } (x, y, z) = (-2, 2, 1).$$

Case 3: $z = -1$.

The equation for \mathcal{F}_λ now implies that $x^2 = -6$. Thus there is no solution when $z = -1$.

6. In the event that the system of equations has more than one solution, compute the value of the objective function at each solution point and take the point(s) that gives the maximum or minimum.

At the point $(x, y, z) = (0, 0, 5)$ the objective function takes value 25.

At the points $(x, y, z) = (2, -2, 1)$ and $(x, y, z) = (-2, 2, 1)$ the objective function takes the value 9.

Thus the points $(x, y, z) = (2, -2, 1)$ and $(x, y, z) = (-2, 2, 1)$ are closest to the origin.