Chapter 2

Singularity theory with a distinguished parameter

2.1 Introduction

In chapter 1.6 the spruce budworm model was introduced

$$\frac{\mathrm{d}x}{\mathrm{d}t} = rx\left(1 - \frac{x}{q}\right) - \frac{x^2}{1 + x^2}.$$

The steady-state solutions of this model were found to be $x = 0$ and the solutions of the equation

$$\mathcal{G} = r\left(1 + x^2\right)\left(1 - \frac{x}{q}\right) - x = 0.$$

When $q = 10$ or $q = 20$ the steady-state diagram, shown in figure 1.10, of this last equation was found to have two limit point bifurcations. When $q = 3$ the steady-state diagram, shown in figure 1.11, did not appear to contain a limit point. We noted that these steady-state diagrams seemed to differ in a 'fundamental' manner. By numerical trial-and-error you have found a critical value of q, q_{cr}, such that if $q > q_{\mathrm{cr}}$ the steady-state diagram is qualitatively the same as those shown in figure 1.10 whereas if $q < q_{\mathrm{cr}}$ the steady-state diagram is qualitatively the same as those shown in figure 1.11.

In chapter 1.7 we raised the following questions. Firstly, is it possible to find the critical value of q without resorting to numerical trial-and-error? Secondly, is it possible to define in a rigorous manner what we mean by steady-state diagrams differing in a 'fundamental' way? Thirdly, given a model, is it possible to find all the different types of steady-state diagrams? For example, we have shown that the spruce budworm model has at least two different types of steady-state diagram — are there any more?

By the end of this chapter we will have answered these questions. The mathematical basis for a detailed analysis of such questions is provided by singularity theory.

2.2 Singularity theory with a distinguished parameter

Many models of physical systems can be reduced to a non-linear implicit scalar equation of the form,

$$G\left(x, \mu, \mathbf{p}\right) = 0. \tag{2.1}$$

This equation contains a state variable (x), a distinguished parameter, (μ), also known as the primary bifurcation parameter, and several secondary bifurcation parameters ($\mathbf{p}$). The choice of a primary bifurcation parameter usually depends upon the physical nature of the underlying problem giving rise to equation (2.1). We will assume that the function G has continuous derivatives of all orders[1] both with respect to the state variable, the primary bifurcation parameter and to the secondary bifurcation parameters where required. Equation (2.1) is often referred to as being the *singularity equation.*

Singularity equations naturally arise when we try to find steady-state solutions of the scalar differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x).$$

The singularity function is given by

$$\mathcal{G} = f(x^*) = 0.$$

For the spruce budworm model we have already obtained the singularity function

$$\mathcal{G} = r\left(1 + x^2\right)\left(1 - \frac{x}{q}\right) - x = 0.$$

(Note that in this equation we have written x rather than x^*. From now on in this chapter we drop the superscript notation.)

2.3 Singularity theory and bifurcation points

In chapter 1.5.2 we considered three archetypical steady-state problems. In chapter 1.5.2.1 we showed that the steady-state diagram for the scalar equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \mu - x^2,$$

has a *limit point* bifurcation at the parameter value $(\mu, x) = (0, 0)$. The steady-state diagram for this system is shown in figure 1.7.

In chapter 1.5.2.2 we showed that the steady-state diagram for the scalar equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \mu x - x^2,$$

has a *transcritical* bifurcation at the parameter value $(\mu, x) = (0, 0)$. The steady-state diagram for this system is shown in figure 1.8.

In chapter 1.5.2.3 we showed that the steady-state diagram for the scalar equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \mu x - x^3,$$

has a *pitchfork* bifurcation at the parameter value $(\mu, x) = (0, 0)$. The steady-state diagram for this system is shown in figure 1.9.

In this section we show how singularity theory can be used to find the location of any limit point, transcritical or pitchfork bifurcation points without having to first construct the steady-state diagram.

2.3.1 The limit point function

Theorem 2.1 (Limit point bifurcation) *Suppose that at the point* $(\mu, x) = (\mu, x_0)$ *the singularity function* $\mathcal{G}$ *satisfies the set of equation*

$$\begin{aligned} \mathcal{G} = \mathcal{G}_x &= 0, \\ \mathcal{G}_{xx} &\neq 0, \\ \mathcal{G}_\mu &\neq 0. \end{aligned}$$

Then a limit point bifurcation occurs at the point (μ, x_0).

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[1] A function that has derivatives of all orders is known as a smooth function.

The conditions $\mathcal{G}_{xx} \neq 0$ & $\mathcal{G}_{\mu} \neq 0$ are known as the non-degeneracy conditions.

Example 2.1 *Show that the steady-state diagram for the differential equation*

$$\frac{dx}{dt} = \mu - x^2$$

has a limit point bifurcation at the point $(\mu, x) = (0, 0)$.

Solution We have

$$\begin{aligned} \mathcal{G} &= \mu - x^2, \\ \mathcal{G}_x &= -2x. \end{aligned}$$

Then

$$\mathcal{G}_x = 0 \Rightarrow x = 0.$$

Hence

$$\mathcal{G} = 0 \Rightarrow \mu = 0.$$

The non-degeneracy conditions are satisfied at the bifurcation point $(\mu, x) = (0, 0)$ as

$$\begin{aligned} \mathcal{G}_{xx} &= -2 \neq 0, \\ \mathcal{G}_{\mu} &= 1 \neq 0. \end{aligned}$$

Thus there is a limit point bifurcation at the point $(\mu, x) = (0, 0)$. □

Question 2.1 *We know that the problem considered in example 2.1 has a limit point at* $(x, \mu) = (0, 0)$ *from chapter 1.5.2.1. We established this result by finding the steady-state solutions as a function of the primary bifurcation parameter and plotting this information in the form of a steady-state diagram. We might therefore ask, what is the point of using 2.1?*

2.3.2 The transcritical bifurcation

Theorem 2.2 (Transcritical bifurcation) *Suppose that at the point* $(\mu, x) = (\mu, x_0)$ *the singularity function* $\mathcal{G}$ *satisfies the set of equation*

$$\begin{aligned} \mathcal{G} = \mathcal{G}_x = \mathcal{G}_{\mu} &= 0, \\ \mathcal{G}_{xx} &\neq 0, \\ \mathcal{G}_{\mu x}^2 - \mathcal{G}_{xx}\mathcal{G}_{\mu\mu} &> 0. \end{aligned}$$

Then a transcritical bifurcation occurs at the point (μ, x_0).

The conditions $\mathcal{G}_{xx} \neq 0$ & $\mathcal{G}_{\mu x}^2 - \mathcal{G}_{xx}\mathcal{G}_{\mu\mu} > 0$ are known as the non-degeneracy conditions.

Example 2.2 *Show that the steady-state diagram for the differential equation*

$$\frac{dx}{dt} = \mu x - x^2$$

has a transcritical bifurcation at the point $(\mu, x) = (0, 0)$.

Solution We have

$$\begin{aligned} \mathcal{G} &= \mu x - x^2, \\ \mathcal{G}_x &= \mu - 2x, \\ \mathcal{G}_\mu &= x. \end{aligned}$$

Then

$$\mathcal{G}_\mu = 0 \Rightarrow x = 0.$$

Hence

$$\mathcal{G}_x = 0 \Rightarrow \mu = 0,$$

and at $(\mu, x) = (0,0)$ we have

$$\mathcal{G} = 0.$$

The non-degeneracy conditions are satisfied at the bifurcation point $(\mu, x) = (0,0)$ as

$$\begin{aligned} \mathcal{G}_{xx} &= -2 \neq 0, \\ \mathcal{G}_{\mu\mu} &= 0, \\ \mathcal{G}_{\mu x} &= 1 \\ \Rightarrow \mathcal{G}_{\mu x}^2 - \mathcal{G}_{xx}\mathcal{G}_{\mu\mu} &= 1 > 0. \end{aligned}$$

Thus there is a limit point bifurcation at the point $(\mu, x) = (0,0)$. □

Note, this example is somewhat 'funny'. We have managed to solve three equations ($\mathcal{G} = \mathcal{G}_x = \mathcal{G}_\mu = 0$) using only two parameters (μ, x). Usually we require four parameters to solve four equations!

2.3.3 The pitchfork bifurcation

Theorem 2.3 (Pitchfork bifurcation) *Suppose that at the point* $(\mu, x) = (\mu, x_0)$ *the singularity function* $\mathcal{G}$ *satisfies the set of equation*

$$\begin{aligned} \mathcal{G} = \mathcal{G}_x = \mathcal{G}_{xx} = \mathcal{G}_\mu &= 0, \\ \mathcal{G}_{xxx} &\neq 0, \\ \mathcal{G}_{\mu x} &\neq 0. \end{aligned}$$

Then a pitchfork bifurcation occurs at the point (μ, x_0).

The conditions $\mathcal{G}_{xxx} \neq 0$ & $\mathcal{G}_{\mu x} \neq 0$ are known as the non-degeneracy conditions. They are sometimes combined to give the equivalent form

$$\mathcal{G}_{xxx} \cdot \mathcal{G}_{\mu x} \neq 0.$$

Example 2.3 *Show that the steady-state diagram for the differential equation*

$$\frac{dx}{dt} = \mu - x^2$$

has a pitchfork bifurcation at the point $(\mu, x) = (0,0)$.

Solution We have

$$\begin{aligned} \mathcal{G} &= \mu x - x^3, \\ \mathcal{G}_x &= \mu - 3x^2, \\ \mathcal{G}_{xx} &= -6x, \\ \mathcal{G}_\mu &= x. \end{aligned}$$

Then

$$\mathcal{G}_{xx} = 0 \Rightarrow x = 0.$$

Hence

$$\mathcal{G}_x = 0 \Rightarrow \mu = 0.$$

and at $(\mu, x) = (0,0)$ we have

$$\mathcal{G} = 0 \quad \& \quad \mathcal{G}_\mu = 0.$$

The non-degeneracy conditions are satisfied at the bifurcation point $(\mu, x) = (0,0)$ as

$$\begin{aligned} \mathcal{G}_{xxx} &= -6 \neq 0, \\ \mathcal{G}_{\mu x} &= 1 \neq 0. \end{aligned}$$

Thus there is a pitchfork bifurcation at the point $(\mu, x) = (0,0)$. □

Note, this example is again somewhat 'funny'. We have managed to solve four equations ($\mathcal{G} = \mathcal{G}_x = \mathcal{G}_{xx} = \mathcal{G}_\mu = 0$) using only two parameters (μ, x). Usually we require four parameters to solve four equations!

2.3.4 Exercises

1. Consider the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x^3 - \mu x + \alpha + \beta x^2.$$

The singularity equation is

$$\mathcal{G} = x^3 - \mu x + \alpha + \beta x^2 = 0.$$

(a) Figure 2.1 shows a steady-state diagram for the singularity equation

$$\mathcal{G} = x^3 - \mu x + \alpha + \beta x^2 = 0.$$

(stability is not indicated).

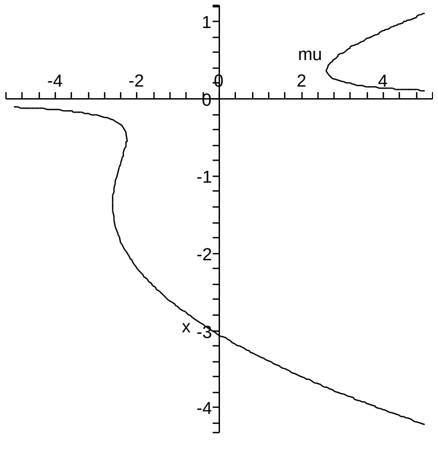


Figure 2.1: Steady-state diagram for the singularity equation $x^3 - \mu x + \alpha + \beta x^2 = 0$. Parameter values: $\alpha = 0.5, \beta = 3.0$.

Write maple code to obtain this figure. Your answer must include your code and the graph that it generates.

(b) Figure 2.1 has three limit points.

  (i) Using theorem 2.1 show that the limit points of the singularity equation

$$\mathcal{G} = x^3 - \mu x + 0.5 + 3x^2 = 0$$

are

$$\begin{aligned}(\mu, x_1) &= (-2.25, -0.5)\,,\\ (\mu, x_2) &= (-2.60, -1.37)\,,\\ (\mu, x_3) &= (2.60, 0.37)\,.\end{aligned}$$

(You may use maple as much as you require.)

  (ii) Show that the required non-degeneracy conditions are satisfied.

(c) Write maple code to obtain figure 2.2. Your answer must include your code and the graph that it generates.

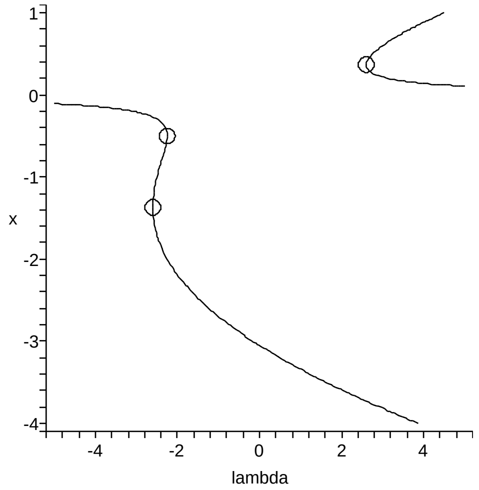


Figure 2.2: Steady-state diagram for the singularity equation $x^3 - \mu x + \alpha + \beta x^2 = 0$. Bifurcation points marked by circles. Parameter values: $\alpha = 0.5, \beta = 3.0$.

(d) Determine the stability of the steady-state solutions shown on figure 2.2. Hence sketch a steady-state diagram that indicates stability.

2. In chapter 1.6 the spruce budworm model was introduced

$$\frac{\mathrm{d}x}{\mathrm{d}t} = rx\left(1 - \frac{x}{q}\right) - \frac{x^2}{1 + x^2}.$$

The steady-state solutions of this model were found to be $x_1 = 0$ and the solutions of the equation

$$\mathcal{G} = r\left(1 + x^2\right)\left(1 - \frac{x}{q}\right) - x = 0.$$

Find the location of the limit points (r, x) for the singularity function $\mathcal{G}$ when

  (i) $q = 10$.
  (ii) $q = 20$.

Check that the appropriate non-degeneracy conditions are satisfied.

2.4 How do steady-state diagrams change in a 'fundamental' way?

In the spruce budworm model the qualitative structure of the steady-state diagram was different when $q = 10$, figure 1.10, from when $q = 3$, figure 1.11. In the former case there are two limit point bifurcations whereas in the latter there were no bifurcation points.

What do we mean by a qualitative, or 'fundamental', change in a steady-state diagram? We do not pursue this matter. However, it turns out that there are only *three* mechanisms by which a steady-state diagram can change in a qualitative manner [27]. These are known as:

1. the *cusp* singularity (or hysteresis point);
2. the *isola* singularity;
3. the *double limit-point* singularity.

Consider a scalar differential equation of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f\left(x, \mu, \alpha\right),$$

where μ is the primary bifurcation parameter and α is a secondary bifurcation parameter. We want to find the values of the secondary bifurcation parameter, α, at which the steady-state diagram changes in a qualitative, or 'fundamental', manner. To do this we introduce the *singularity function* $\mathcal{G}$ defined by

$$\mathcal{G}\left(x, \mu, \alpha\right) = f\left(x, \mu, \alpha\right) = 0.$$

In this section we give the defining conditions for the cusp, isola and double limit-point singularities in terms of the singularity function $\mathcal{G}\left(x, \mu, \alpha\right)$.

2.4.1 The cusp singularity

Theorem 2.4 (Cusp singularity) *Suppose that at the point* $(\mu, x\alpha) = (\mu, x_0, \alpha_0)$ *the singularity function* $\mathcal{G}\left(\mu, x, \alpha\right)$ *satisfies the equations*

$$\begin{aligned} G = G_x = G_{xx} &= 0, \\ G_\mu &\neq 0. \\ G_{xxx} &\neq 0. \end{aligned} \tag{2.2}$$

Then a cusp singularity, or hysteresis point, occurs at the point (μ, x_0, α_0).

Typically when the cusp curve is crossed a hysteresis loop appears or disappears in the steady-state diagram as two limit points appear or disappear, as is shown in figure 2.3.

We can think of the cusp singularity as being a limit-point bifurcation in which the non-degeneracy condition $\mathcal{G}_{\mu\mu} \neq 0$ is violated and is replaced by the condition $\mathcal{G}_{\mu\mu\mu} = 0$.

Example 2.4 *Find the value(s) of* α *for which the singularity function*

$$\mathcal{G} = x^3 - \mu + \alpha x = 0$$

has a cusp singularity.

Solution We have

$$\begin{aligned} \mathcal{G} &= x^3 - \mu + \alpha x = 0, \\ \mathcal{G}_x &= 3x^2 + \alpha, \\ \mathcal{G}_{xx} &= 6x. \end{aligned}$$

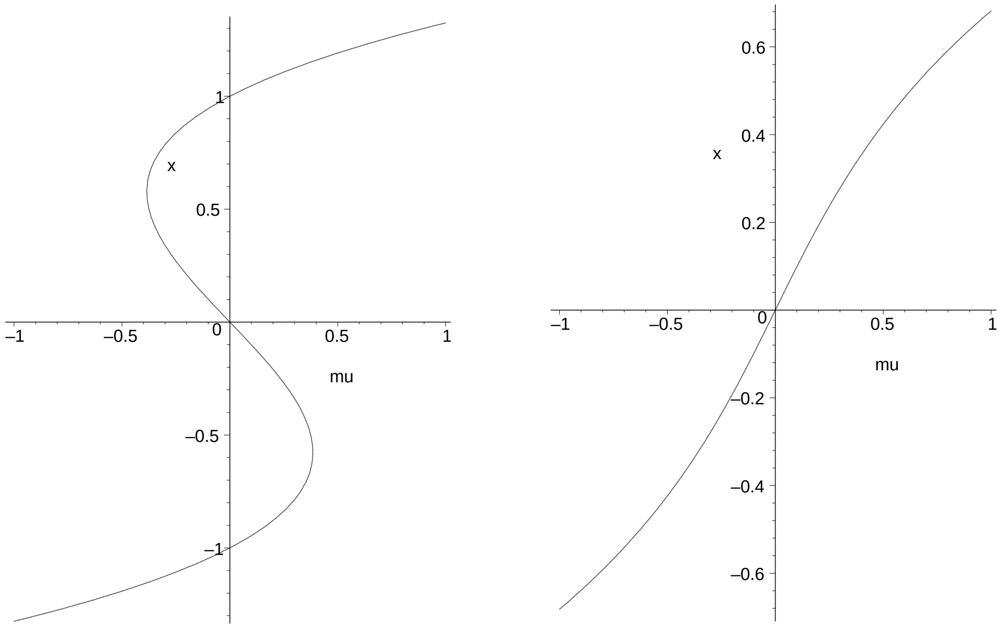


Figure 2.3: The disappearance/appearance of a hysteresis loop in a steady-state diagram due to a cusp singularity.

Then

$$\begin{aligned} \mathcal{G}_{xx} &= 0 \Rightarrow x = 0, \\ \mathcal{G}_x &= 0 \Rightarrow \alpha = 0, \\ \mathcal{G} &= 0 \Rightarrow \mu = 0. \end{aligned}$$

The non-degeneracy conditions are satisfied at the cusp point $(\mu, x, \alpha) = (0, 0, 0)$ as

$$\begin{aligned} \mathcal{G}_{xxx} &= 6 \neq 0, \\ \mathcal{G}_\mu &= -1 \neq 0. \end{aligned}$$

Thus there is a cusp singularity at the point $(\mu, x, \alpha) = (0, 0, 0)$. □

We therefore know that the steady-state diagram when $\alpha < 0$ is different in a 'fundamental' way from that when $\alpha > 0$. In one of these cases the steady-state diagram will look qualitatively the same as figure 2.3 (a) whereas in the other case the steady-state diagram will look qualitatively the same as figure 2.3 (b). Which ones is which? Although singularity theory provide a tool to answer this question, in practice it is simpler to plot the steady-state diagrams for $\alpha < 0$ and $\alpha > 0$.

Example 2.5 *For the singularity equation*

$$\mathcal{G} = x^3 - \mu x + \alpha + \beta x^2.$$

find an equation in the (α, β) *plane along which the cusp singularity occurs.*

Solution We have

$$\begin{aligned} \mathcal{G} &= x^3 - \mu x + \alpha + \beta x^2 = 0, \\ \mathcal{G}_x &= 3x^2 - \mu + 2\beta x, \\ \mathcal{G}_{xx} &= 6x + 2\beta. \end{aligned}$$

Then

$$\begin{aligned} \mathcal{G}_{xx} &= 0 \Rightarrow x = -\frac{\beta}{3}, \\ \mathcal{G}_{x} &= 0 \Rightarrow \mu = -\frac{\beta^2}{3}, \\ \mathcal{G} &= 0 \Rightarrow \alpha = \frac{\beta^3}{27}. \end{aligned}$$

The non-degeneracy conditions are satisfied along the curve

$$(\mu, x, \alpha, \beta) = \left(-\frac{\beta^2}{3}, -\frac{\beta}{3}, \frac{\beta^3}{27}, \beta\right)$$

as

$$\begin{aligned} \mathcal{G}_{xxx} &= 6 \neq 0, \\ \mathcal{G}_{\mu} &= -x = \frac{\beta}{3} \neq 0 \qquad \text{provided that} \qquad \beta \neq 0. \end{aligned}$$

Thus there is a cusp singularity along the curve

$$(\mu, x, \alpha, \beta) = \left(-\frac{\beta^2}{3}, -\frac{\beta}{3}, \frac{\beta^3}{27}, \beta\right)$$

provided that $\beta \neq 0$. Thus the required equation in the (α, β) plane is

$$\alpha = \frac{\beta^3}{27}$$

subject to the restriction that $\beta \neq 0$. □

2.4.2 The isola singularity

Theorem 2.5 (Isola singularity) *Suppose that at the point* $(\mu, x, \alpha) = (\mu, x_0, \alpha_0)$ *the singularity function* $\mathcal{G}(\mu, x, \alpha)$ *satisfies the equations*

$$\begin{aligned} G = G_x = G_\mu &= 0, \\ G_{xx} &\neq 0, \\ G_{xx}G_{\mu\mu} - (G_{x\mu})^2 &\neq 0. \end{aligned} \tag{2.3}$$

Then an isola singularity occurs at the point (μ, x_0, α_0).

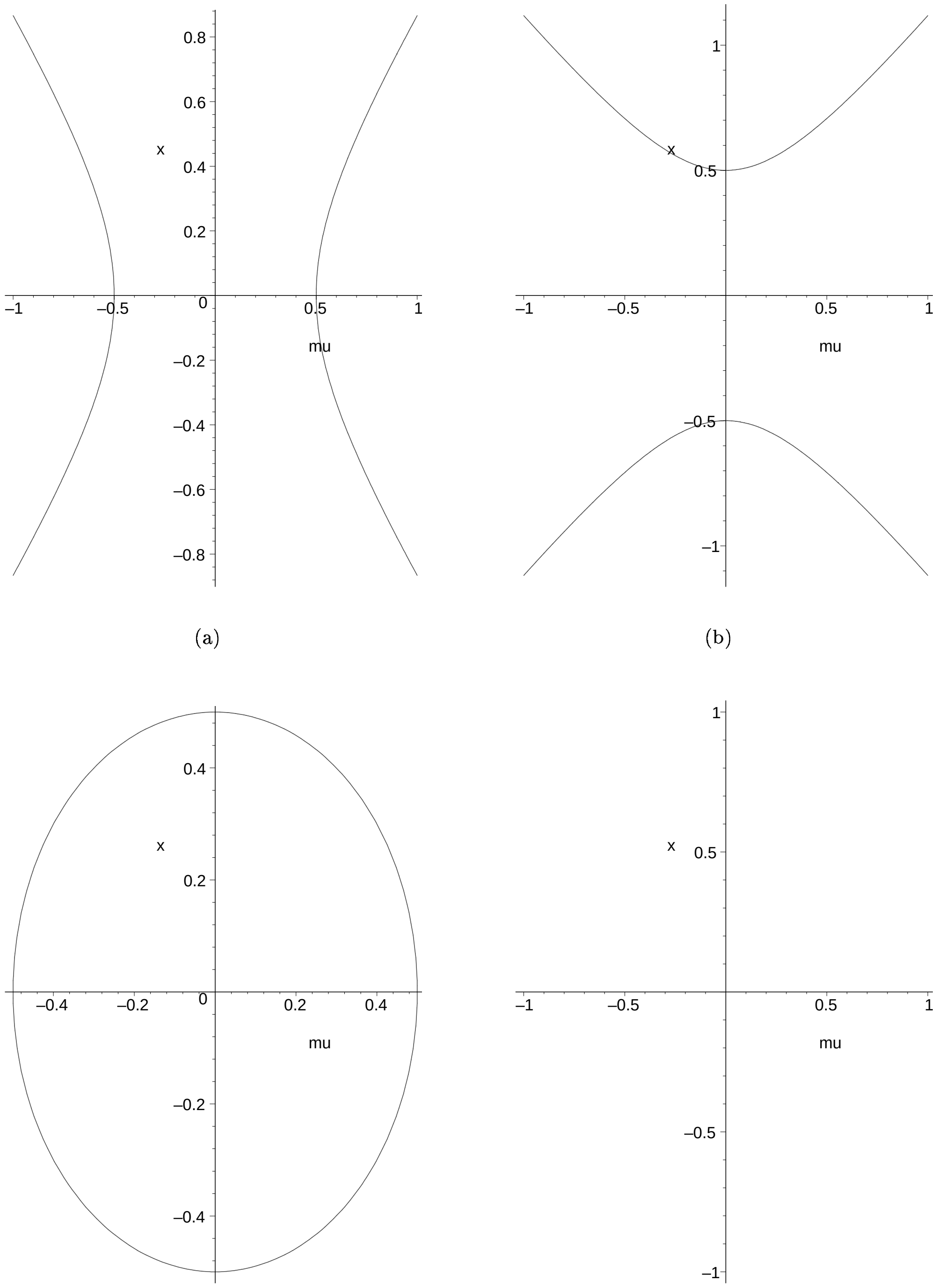
When the isola singularity is crossed two limit points appear or disappear. Two types of behaviour may occur. In the first case, illustrated in figures 2.4 (a & b), the steady-state diagrams separate locally into two isolated curves. We call this transition the transcritical variety of the isola singularity. In the second second, illustrated in figures 2.4 (c & d), an isolated branch of connected solutions appears or disappears. We call this transition the isola variety of the isola singularity.

The isola singularity is very closely related to the transcritical bifurcation. The only difference is that in the former we have $G_{xx}G_{\mu\mu} - (\mathcal{G}_{x\mu})^2 \neq 0$ whereas in the latter we have $G_{xx}G_{\mu\mu} - (\mathcal{G}_{x\mu})^2 > 0$.

The type of transition that occurs depends upon the sign of the function

$$G_{xx}G_{\mu\mu} - G_{x\mu}^2,$$

evaluated at the isola singularity. In practice it is simpler to plot the steady-state diagrams for $\alpha < 0$ and $\alpha > 0$ and deduce from these which type of transition has occurred.



(a) (b)

(c) (d)

Figure 2.4: The separation of the steady-state diagram into isolated curves due to an isola singularity (transcritical variety) (a-b) and the appearance/disappearance of an isolated branch of connected solutions due to an isola singularity (isola variety) (c-d).

Example 2.6 *Find the value(s) of α for which the singularity function*

$$\mathcal{G} = x^2 - \mu^2 + \alpha = 0$$

has an isola singularity.

Solution We have

$$\begin{aligned}\mathcal{G} &= x^2 - \mu^2 + \alpha = 0,\\ \mathcal{G}_x &= 2x,\\ \mathcal{G}_\mu &= -2\mu,\end{aligned}$$

Then

$$\begin{aligned}\mathcal{G}_\mu &= 0 \Rightarrow \mu = 0,\\ \mathcal{G}_x &= 0 \Rightarrow x = 0,\\ \mathcal{G} &= 0 \Rightarrow \alpha = 0.\end{aligned}$$

The non-degeneracy conditions are satisfied at the isola point $(\mu, x, \alpha) = (0, 0, 0)$ as

$$\begin{aligned}\mathcal{G}_{xx} &= 2 \neq 0,\\ \mathcal{G}_{\mu\mu} &= -2,\\ \mathcal{G}_{x\mu} &= 0,\\ \Rightarrow G_{xx}G_{\mu\mu} - (G_{x\mu})^2 &= 4 \neq 0.\end{aligned}$$

Thus there is an isola singularity at the point $(\mu, x, \alpha) = (0, 0, 0)$.

Example 2.7 *For the singularity equation*

$$\mathcal{G} = x^3 - \mu x + \alpha + \beta x^2.$$

find an equation in the (α, β) plane along which the isola singularity occurs.

Solution We have

$$\begin{aligned}\mathcal{G} &= x^3 - \mu x + \alpha + \beta x^2 = 0,\\ \mathcal{G}_x &= 3x^2 - \mu + 2\beta x,\\ \mathcal{G}_\mu &= -x.\end{aligned}$$

Then

$$\begin{aligned}\mathcal{G}_\mu &= 0 \Rightarrow x = 0,\\ \mathcal{G}_x &= 0 \Rightarrow \mu = 0,\\ \mathcal{G} &= 0 \Rightarrow \alpha = 0.\end{aligned}$$

The non-degeneracy conditions are satisfied along the curve

$$(\mu, x, \alpha, \beta) = (0, 0, 0, \beta)$$

as

$$\begin{aligned}\mathcal{G}_{xx} &= 6x + 2\beta = 2\beta \neq 0, \quad \text{provided that} \quad \beta \neq 0,\\ \mathcal{G}_{\mu\mu} &= 0,\\ \mathcal{G}_{x\mu} &= -1,\\ \Rightarrow \mathcal{G}_{xx}\mathcal{G}_{\mu\mu} - (\mathcal{G}_{x\mu})^2 &= -1 \neq 0.\end{aligned}$$

Thus there is an isola singularity along the curve

$$(\mu, x, \alpha, \beta) = (0, 0, 0, \beta)$$

provided that $\beta \neq 0$. Thus the required equation in the (α, β) plane is

$$\alpha = 0$$

subject to the restriction that $\beta \neq 0$. □

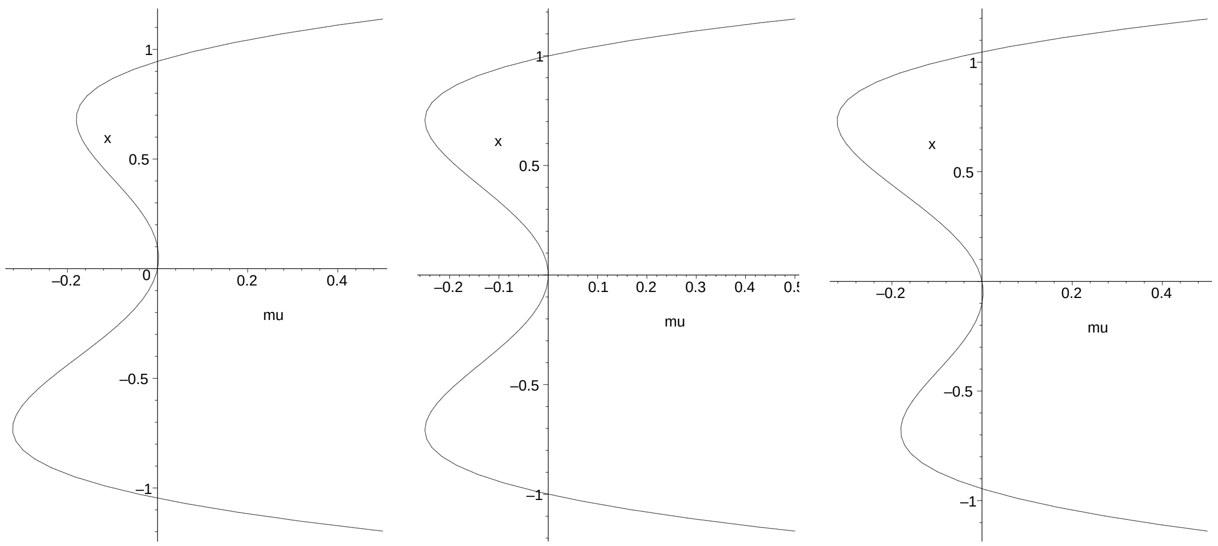
2.4.3 The double limit-point variety

Theorem 2.6 (Double limit-point singularity) *Suppose that at the points* $(\mu, x, \alpha) = (\mu, x_0, \alpha_0)$ *and* $(\mu, x, \alpha) = (\mu, x_1, \alpha_0)$ *the singularity function* $\mathcal{G}(\mu, x\alpha)$ *satisfies the equations*

$$\begin{aligned} G(x_0, \mu, \alpha) = G(x_1, \mu, \alpha) &= 0. \\ G_x(x_0, \mu, \alpha) = G_x(x_1, \mu, \alpha) &= 0. \\ G_{xx}(x_0, \mu, \alpha) \neq 0 \quad G_{xx}(x_1, \mu, \alpha) &\neq 0, \\ G_\mu(x_0, \mu, \alpha) \neq 0 \quad G_\mu(x_1, \mu, \alpha) &\neq 0, \\ x_0 \neq x_1. \end{aligned} \tag{2.4}$$

Then a double limit-point singularity occurs at the points (μ, x_0, α_0) *and* (μ, x_1, α_0).

At a double-limit point two limit points, at x_0 and x_1, occur at the same value of the distinguished parameter. (The non-degeneracy conditions for a limit point bifurcation must also hold at the points (μ, x_i).) As the the double limit point singularity is crossed the number of limit points does not change. What does change is the relative position of these limit points. This transition is shown in figure 2.5. Figure 2.5 (b) corresponds to parameter values at which a double-limit point singularity occurs: two of the limit points in this figure occur at the same value of the primary bifurcation parameter (μ) but for different values of the state-variable (x).



(a)

(b) The double-limit point singularity.

(c)

Figure 2.5: The relative changing of position of two limit-points as a double-limit point singularity is crossed in a bifurcation diagram.

Example 2.8 *For the singularity equation*

$$\mathcal{G} = x^3 - \mu x + \alpha + \beta x^2.$$

find an equation in the (α, β) *plane along which the double limit point singularity occurs.*

Solution When the double limit point singularity occurs there is a value of the distinguished parameter, or primary bifurcation parameter, (here μ) for which two distinct limit points occur. This means that there is a neighbourhood of the critical value of μ in which the singularity function has *four* solutions, see figure 2.5. A cubic equation can not have four solutions. Thus for the given system the double limit point singularity can not occur. □

For all but the simplest problem there is very little hope of finding the location of a double-limit point by hand. We must expect to use numerical techniques to find the double limit-point singularity.

Mention quartic fold singularity here and rewrite this section when the quartic fold singularity has been added.

2.4.4 Exercises

1. Consider the singularity function
$$\mathcal{G} = x^3 - \mu + \alpha x = 0.$$
    (a) Show that there is a critical value of α, α_{cr}, such that a cusp singularity occurs when $\alpha = \alpha_{\text{cr}}$.
    (b) Can either the isola or double limit-point singularities occur in this model?
    (c) By considering steady-state diagrams for $\alpha < \alpha_{\text{cr}}$ and $\alpha > \alpha_{\text{cr}}$ explain how the steady-state diagram depends upon the value of α. Provide appropriate steady-state diagrams (do not show stability).

2. Consider the singularity function
$$\mathcal{G} = x^2 - \mu^2 + \alpha = 0.$$
    (a) Show that there is a critical value of α, α_{cr}, such that an isola singularity occurs when $\alpha = \alpha_{\text{cr}}$.
    (b) Can either the cusp or double limit-point singularities occur in this model?
    (c) By considering steady-state diagrams for $\alpha < \alpha_{\text{cr}}$ and $\alpha > \alpha_{\text{cr}}$ explain how the steady-state diagram depends upon the value of α. Provide appropriate steady-state diagrams (do not show stability).
    (d) Is this an example of the 'transcritical singularity' or the 'isola singularity'?

3. Consider the singularity function
$$\mathcal{G} = x^2 + \mu^2 + \alpha = 0$$
    (a) Show that there is a critical value of α, α_{cr}, such that an isola singularity occurs when $\alpha = \alpha_{\text{cr}}$.
    (b) Can either the cusp or double limit-point singularities occur in this model?
    (c) By considering steady-state diagrams for $\alpha < \alpha_{\text{cr}}$ and $\alpha > \alpha_{\text{cr}}$ explain how the steady-state diagram depends upon the value of α. Provide appropriate steady-state diagrams (do not show stability).
    (d) Is this an example of the 'transcritical singularity' or the 'isola singularity'?

2.5 Constructing a bifurcation diagram

2.5.1 Introduction

In this section we answer the third point that was raised in section 2.1. Given a model, is it possible to find all the qualitatively different types of steady-state diagrams? In practical problems it is important to be able to specify all possible steady-state diagrams and to identify where they occur in parameter space. Why is it important? It is important because we want to know how a change in a design or operating parameter will change the structure of the steady-state diagram.

Given what we have already learnt we rephrase this question in a slightly different way. Given a singularity function,

$$\mathcal{G}\left(x, \mu, \mathbf{p}\right) = 0, \tag{2.5}$$

it is possible to find all the qualitatively different types of steady-state diagrams?

In equation 2.5 μ is, the distinguished parameter, or more commonly, the bifurcation parameter, x is a scalar variable, and $\mathbf{p}$ is the set of secondary bifurcation parameters. Different values for the secondary bifurcation parameters **p** may produce qualitatively different steady-state diagrams. For practical problems it is often very important to know the values of the parameters **p** for which different steady-state diagrams occur.

To illustrate the variation in steady-state diagrams as the secondary bifurcation parameters (**p**) are changed compare figures 2.1 & 2.6, which both show a steady-state diagram singularity equation

$$\mathcal{G} = x^3 - \mu x + \alpha + \beta x^2 = 0.$$

The steady-state diagram shown in 2.1, with $\alpha = 0.5$ and $\beta = 3.0$, is qualitatively different from that shown in figure 2.6, where $\alpha = 0.5$ and $\beta = 1.0$. How many other steady-state diagrams does this singularity function have?

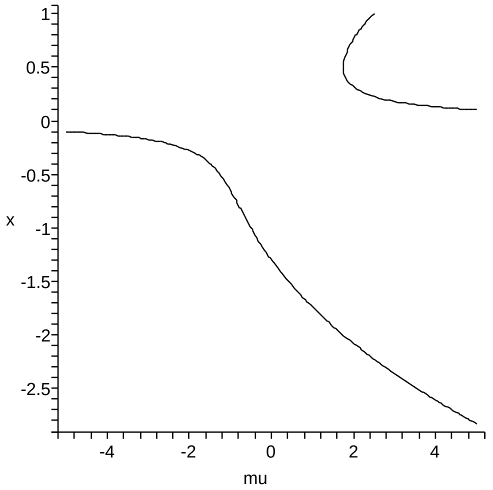


Figure 2.6: Steady-state diagram for the singularity equation $x^3 - \mu x + \alpha + \beta x^2 = 0$. Parameter values: $\alpha = 0.5, \beta = 1.0$.

Question 2.2

1. *Using theorem 2.1 (and maple) find the bifurcation points of the singularity equation*

$$\mathcal{G} = x^3 - \mu x + 0.5 + x^2 = 0.$$

2. *In what ways are the steady-state diagrams shown in figures 2.1 & 2.6 different?*
3. *As β has decreased from $\beta = 3.0$, figure 2.1, to $\beta = 1.0$, figure 2.6, what kind of singularity has occurred?*

Definition 2.1 (Bifurcation diagram) *A figure which shows divides the parameter space* **p** *into regions with qualitatively different kinds of steady-state diagrams is known as a* ***bifurcation diagram****.*

A bifurcation diagram may be in any number of dimensions depending upon the number of parameters in **p**. For example, an equation

$$G(x, \mu, \alpha) = 0$$

will have a one-dimensional bifurcation diagram — consisting of a line. The equation

$$G(x, \mu, \alpha, \beta) = 0$$

will have a two-dimensional bifurcation diagram in the $\alpha - \beta$ plane.

If one or two parameters then why not bifurcation diagrams with three, four, or more secondary bifurcation parameters? Mathematically, there is no reason to limit the number of secondary bifurcation parameters. A problem with two secondary bifurcation parameters produces a bifurcation diagram that is drawn in a plane.

This is both easy to construct and easy to understand. A problem with three secondary bifurcation parameters produces a bifurcation diagram in three dimensions. This is more difficult to draw, and much more difficult to understand — even with software visualisation. It is impossible to draw bifurcation diagrams for problems containing four, or more, secondary bifurcation parameters. In practice two secondary bifurcation parameters often suffices.

We discuss in the next section how to a construct bifurcation diagram.

2.5.2 The method

A bifurcation diagram will generally contain curves, points in one dimension, which divide it into regions. Crossing from one-region to another will cause the steady-state diagram to change in a 'qualitative' manner. Obviously it is important to define what one means by a 'qualitative' change in a steady-state diagram.

In section 2.4 we stated there are only three mechanisms by which a steady-state diagram can change in a qualitative manner. These are:

1. the *cusp* (or hysteresis) singularity;
2. the *isola* singularity; and,
3. the *double limit-point* singularity.

The defining conditions for these singularities were stated in sections 2.4.1, 2.4.2, & 2.4.3 respectively.

A qualitative change in a steady-state diagram occurs if and only if the secondary bifurcation parameters $\mathbf{p}$ cross the boundaries that define these transitions [27]. Bifurcation diagram are constructed by determining the locus of these three curves in parameter space $\mathbf{p}$. This method divides the parameter space into regions, each corresponding to a different steady-state diagram of the problem $G = 0$. Thus once the parameter space has been divided into regions all that remains is to determine one steady-state diagram in each region bounded by these surfaces.

Example 2.9 *Construct the bifurcation diagram for the singularity function*

$$\mathcal{G} = x^3 - \mu x + \alpha + \beta x^2 = 0.$$

Solution To construct the bifurcation diagram we must determine the locus in the (α, β) plane of:

1. the cusp singularity;
2. the isola singularity, and,
3. the double limit point singularity

From example 2.5 we know that the cusp singularity occurs along the curve

$$\alpha = \frac{\beta^3}{27}.$$

From example 2.7 we know that the isola singularity occurs along the curve

$$\alpha = 0.$$

From example 2.8 we know that the double limit point singularity does not occur in this problem.

Thus the bifurcation diagram for this problem contains the lines $\alpha = 0$ and $\alpha = \frac{\beta^3}{27} = 0$. As shown in figure 2.7 these divide the $\alpha - \beta$ plane into four regions. Thus there are four, and only four, qualitatively different steady-state diagrams for this problem. □

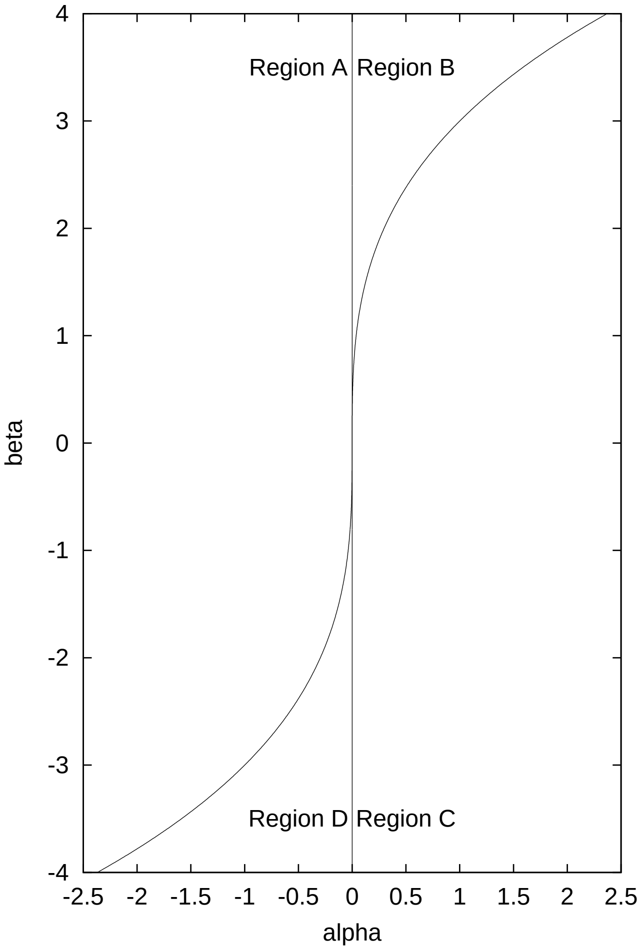


Figure 2.7: Bifurcation diagram for the problem $\mathcal{G} = x^3 - \mu x + \alpha + \beta x^2 = 0$. The steady-state diagrams for regions B& C are shown in figures 2.1 & 2.6 respectively.

Question 2.3

1. *Obtain steady-state diagrams for regions A & D of figure 2.7. How many bifurcation points are there on there on each steady-state diagram?*

2. *Starting in region A explain how the steady-state diagram changes as you move from region A into region B, from region B into region C, from region C into region D and from region D into region A.*

3. *From figure 2.7 the point* $(\alpha, \beta) = (0, 0)$ *is special because the cusp singularity curve,* $\alpha = \dfrac{\beta^3}{27}$ *and the isola singularity curve, the line* $\alpha = 0$*, intersect at this point.*

   (a) *Find the steady-state diagram at the point* $(\alpha, \beta) = (0, 0)$.

   (b) *What kind of bifurcation occurs in this steady-state diagram?*

2.5.3 Questions

In the following $\mathcal{G}$ is a steady-state equation, x is a state variable, μ is the primary bifurcation parameter and α & β are secondary bifurcation parameters.

1. For each of the following functions construct the bifurcation diagram in the $\alpha - \beta$ plane and provide representative steady-state diagrams.

(a) $\mathcal{F} = x^2 - \mu^3 + \alpha + \beta\mu = 0$.
Show that there is a point on the isola locus where the degeneracy condition

$$G_{xx}G_{\mu\mu} - G_{x\mu}^2 \neq 0$$

is violated. Mark the location of this point on your bifurcation diagram.

(b) $\mathcal{F} = x^4 - \mu + \alpha x + \beta x^2 = 0$.
Show that there is a point on the cusp locus where the degeneracy condition

$$G_{xxx} \neq 0$$

is violated. Mark the location of this point on your bifurcation diagram.

2. The spruce budworm model is given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = rx\left(1 - \frac{x}{q}\right) - \frac{x^2}{1+x^2}.$$

Find all the steady-state diagrams for this problem when the primary bifurcation parameter is r. You may assume that $r > 0$ and $q > 0$.

3. (This question is based upon work presented in a research paper and is likely to be non-trivial!)
Consider the singularity function [10, pages 1616 & 1617]

$$\mathcal{G}(x, \mu, \alpha_1, \alpha_2, \alpha_3) = x^5 - \alpha_3 x^3 - \alpha_2 x^2 - \alpha_1 x - \mu$$

(a) Find the parameter values at which the butterfly catastrophe occurs for this singularity function.
(b) Show that no higher-order singularity can occur.
(c) Show that the hysteresis singularity can be parameterised by

$$\begin{aligned}\alpha_1 &= 3\alpha_3 x^2 - 15x^4,\\ \alpha_2 &= 10x^3 - 3\alpha_3 x,\end{aligned}$$

where the 'parameter' x varies over the range $-\infty < x < \infty$.
(d) Show that the double limit-point singularity can be parameterised by

$$\begin{aligned}\alpha_1 &= \frac{t\left(4t^2 + 7t + 4\right)}{\left(3t^2 + 4t + 3\right)^2}\alpha_3^2,\\ \alpha_2^2 &= \frac{4\left(1+t\right)^2\left(1 + 3t + t^2\right)^2}{\left(3t^2 + 4t + 3\right)^3}\alpha_3^3,\end{aligned}$$

with $-\infty < t < \infty$ being a parameter. cases $\alpha_3 < 0$ and $\alpha_3 > 0$.
(e) Plot the cusp hysteresis curve in the (α_1, α_2) plane for the case $\alpha_3 < 0$.
  (a) Into how many regions does the cusp curve divide the bifurcation plane?
  (b) How many limit points are there in each region?
  (c) Can the double limit-point singularity occur in this case? If no, explain why not. If yes, add the double limit-point curve to your bifurcation diagram.
  (d) How many generic steady-state diagrams are there?
  (e) Provide an illustrative steady-state diagram for each region.
(f) Plot the cusp hysteresis curve in the (α_1, α_2) plane for the case $\alpha_3 > 0$.
  (i) Into how many regions does the cusp curve divide the bifurcation plane?
  (ii) How many limit points are there in each region?
  (iii) Can the double limit-point singularity occur in this case? If no, explain why not. If yes, add the double limit-point curve to your bifurcation diagram.
  (iv) How many generic steady-state diagrams are there?
  (v) Provide an illustrative steady-state diagram for each region.

2.6 Some background on the pure mathematics of singularity theory

In this section we give a little taste of the mathematics behind singularity theory. I will call this 'mathematical singularity theory'. If you can't get your head around this section then don't panic! You don't need to understand it to be able to apply singularity theory to construct bifurcation diagrams. And that is what you need to do. Familiarity with the concepts of this section will only be useful to you if you are going to read papers on mathematical singularity theory.

Consider the singularity equation

$$\mathcal{G} = (x, \mu, \mathbf{p}) \, .$$

We have seen that singularities occur when the function $\mathcal{G}$ and certain of its derivatives vanish. For instance, the cusp singularity occurs when

$$\begin{aligned} \mathcal{G} = \mathcal{G}_x = \mathcal{G}_{xx} &= 0, \\ \mathcal{G}_{xxx}\mathcal{G}G_\mu &\neq 0, \end{aligned}$$

Singularities happen when the secondary bifurcation parameters, $\mathbf{p}$, take certain values, say $\mathbf{p}_0$. For values of $\mathbf{p}$ near the critical value $\mathbf{p}_0$ the singularity equation exhibits qualitatively different steady-state diagrams — it is this mechanism that allows us to find all possible steady-state diagrams for the singularity function $\mathcal{G}$.

One of the questions that mathematical singularity theory was developed to answer is to establish which types of steady-state diagrams exist for different types of singularities. We have learnt of three singularities: the cusp, isola and double limit-point; there are others.

It is now time to introduce some terminology.

Definition 2.2 (Contact Equivalent) *Two steady-state diagrams* $\mathcal{F}(x, \mu)$ *and* $\mathcal{G}(y, \lambda)$ *are said to be contact equivalent if the steady-state diagram of one can be transformed into the other without changing any of its qualitative features in the neighbourhood of the origin by using a smooth and invertible change of coordinates, possible combined with a scaling factor. The phrase 'without changing its qualitative features' means that the number and order of the steady-state solutions* (x) *of the function* $\mathcal{F}$ *for any fixed value of* μ *are the same as the number and order of the steady-state solutions of the function* $\mathcal{G}(y)$ *for the corresponding* λ *[27].*

Why is this concept useful? Suppose that the function $\mathcal{H}$ is a horribly complicated non-linear function whilst the function $\mathcal{P}$ is a simple polynomial. If the functions $\mathcal{H}$ and $\mathcal{P}$ are contact equivalent then the qualitative features of the steady-state diagram of $\mathcal{H}$ can be found by studying those of the function $\mathcal{P}$. As $\mathcal{P}$ is a much simpler function, being a polynomial, it is much easier to study the steady-state diagrams of $\mathcal{P}$ than the original function $\mathcal{H}$.

Consider a function satisfying the requirements of a cusp singularity

$$\begin{aligned} \mathcal{G} = \mathcal{G}_x = \mathcal{G}_{xx} &= 0, \\ \mathcal{G}_\mu\mathcal{G}_\mu &\neq 0, \end{aligned} \tag{2.6}$$

at the singular point $(x, \mu) = (0, 0)$.

A key topic in mathematical singularity theory is to establish the 'best' way to approximate the function $\mathcal{G}$ near the singular point. In the next part of this section we'll discuss in a loose way the meaning of the word 'best' in this context. The simplest function to satisfy 2.6 is

$$\mathcal{G}_1 = x^3 - \mu.$$

However, the steady-state diagram for the function $\mathcal{G}_1$ is *unstable* for the problem 2.6. What does it mean to say that the steady-state diagram of the function $\mathcal{G}_1$ is unstable?

Question 2.4 *Determine the steady-state diagram for the function* $\mathcal{G}_1$.

In chapter 1.4 we introduced the concept of stability for a first-order differential equation. A steady-state solution was stable if when it is perturbed by a 'small' amount we did not qualitatively change the long-term behaviour of the system, i.e. the system returns to the steady-state. Stability is being used in a related sense here.

Definition 2.3 (Stability of a steady-state diagram) *The steady-state diagram of the function* $\mathcal{G}_1$ *is said to be stable if when we change the function* $\mathcal{G}_1$ *by a 'small' amount we do not change the qualitative features of the steady-state diagram.*

Example 2.10 (The steady-state diagram of the function $\mathcal{G}_1$ is unstable) *Consider the function* $\mathcal{G}_2$ *defined by*

$$\mathcal{G}_2 = x^3 - \alpha x - \mu = 0, \qquad |\alpha| \ll 1.$$

This function is 'close' to the function $\mathcal{G}_1$*, being equal when* $\alpha = 0$*. However, the steady-state diagrams of the function* $\mathcal{G}_2$ *assume different forms depending upon the sign of the secondary bifurcation parameter* α *and this is true no matter how small this parameter is.*

Thus the steady-state diagram of $\mathcal{G}_1$ *is unstable because we can perturb the function* $\mathcal{G}_1$ *by an arbitrary amount and obtain qualitatively different steady-state diagrams.*

Question 2.5 *Determine the bifurcation diagram for the singularity equation*

$$\mathcal{G}_2 = x^3 - \alpha x - \mu = 0, \qquad |\alpha| \ll 1.$$

1. *Show that the function* $\mathcal{G}_2$ *has two qualitatively different steady-state diagrams corresponding to* $\alpha < 0$ *and* $\alpha > 0$*;*
2. *Compare these steady-state diagrams to the steady-state diagram for the function* $\mathcal{G}_1$*, which corresponds to the case* $\alpha = 0$ *in* $\mathcal{G}_2$*.*

It can be shown that each of the steady-state diagrams for the function $\mathcal{G}_2$ is *stable* for $\alpha \neq 0$. Is the function $\mathcal{G}_2$ the simplest stable function that represents problem 2.6? To develop an answer to this question we need to introduce a few more ideas.

Definition 2.4 (Unfolding of a function) *An unfolding of a function* $\mathcal{G}$ *is a smoothly parameterized family of functions which includes* $\mathcal{G}$*.*

Example 2.11 (Example of an unfolding of a function) *The family of functions represented by* $\mathcal{G}_2$ *is an unfolding of* $\mathcal{G}_1$*.*

Definition 2.5 (Versal unfolding) *A versal unfolding of* $\mathcal{G}$ *is an unfolding which contains, up to contact equivalence, all the possible small perturbations of the function* $\mathcal{G}$*.*

Example 2.12 (A function that is an unfolding but not a versal unfolding) *The function*

$$\mathcal{G}_3 = x^3 + \alpha^2 x - \lambda$$

is an unfolding of the function $\mathcal{G}_1$ *but it is not a versal unfolding. The reason for this is that the function* $\mathcal{G}_3$ *does not exhibit the same qualitative steady-state diagrams that the unfolding function* $\mathcal{G}_2$ *does.*

Question 2.6 *Determine all qualitatively different steady-state diagrams for the function* $\mathcal{G}_3$*. Show that they do not include all of those found in the unfolding function* $\mathcal{G}_2$*.*

Definition 2.6 (Universal unfolding) *A universal unfolding of the function* $\mathcal{G}$ *is a versal unfolding with the minimum number of secondary bifurcation parameters.*

Definition 2.7 (Codimension) *The number of secondary bifurcation parameters in a universal unfolding of the function* $\mathcal{G}$ *is called the codimension of* $\mathcal{G}$*.*

Example 2.13 (Example of a universal unfolding and of codimension) *It can be shown that the function* $\mathcal{G}_2$ *is a universal unfolding of the function* $\mathcal{G}_1$*. The codimension of* $\mathcal{G}_2$ *is one.*

We have now answered the question which is the 'best' function to model problem 2.6: $\mathcal{G}_2$.

The function

$$\mathcal{G}_4 = x^3 - \beta x^2 - \alpha x - \lambda = 0$$

is a a versal unfolding of the function $\mathcal{G}_1$. However, it has codimension two and therefore can not be a universal unfolding of the function $\mathcal{G}_1$.

Question 2.7 *Find all the qualitative different steady-state diagrams of the function* $\mathcal{G}_4$*.*

The point in the secondary bifurcation parameter space $\mathbf{p}_0$ is a singular point if in the vicinity of $\mathbf{p}_0$ the function $\mathcal{G}$ exhibits qualitatively different stable steady-state diagrams. Singular points are characterised by the vanishing of the singularity function and some of its derivatives. Two questions that are of interest in mathematical singularity theory are

1. What combination of derivatives are required to vanish for the point $\mathbf{p}_0$ to be a singular point?
2. For a given singular point find all qualitatively stable steady-state diagrams.

It turns out, and this is useful for the mathematical analysis, that for any singular point there is polynomial function that is contact equivalent to the function $\mathcal{G}$. Thus mathematical singularity theory reduces, after a considerable amount of work, to the study of polynomial equations.

The universal unfolding for some of the singularities that we have already come across are provided in table 2.1.

Singularity	Universal Unfolding
Cusp	$\mathcal{G} = x^3 - \mu + \alpha x = 0.$
Isola (isola case)	$\mathcal{G} = x^2 + \mu^2 + \alpha = 0$
Isola (transcritical case)	$\mathcal{G} = x^2 - \mu^2 + \alpha = 0.$
Pitchfork	$\mathcal{G} = x^3 - \mu x + \alpha + \beta x^2 = 0$

Table 2.1: Examples of universal unfoldings for common singularities.

2.7 Not just the cusp, isola and double limit-point: Higher order singularities

In this section we discuss some higher-order singularities.

2.7.1 How many steady-state solutions does that singularity function in the window have?

Theorem 2.7 *Consider the singularity function*

$$\mathcal{G}(x, \mu, \mathbf{p}) = 0,$$

which satisfies

$$\begin{aligned}\mathcal{G}\left(0,0,\mathbf{p}_0\right) &= 0,\\ \frac{\partial^i \mathcal{G}}{\partial x^i}\left(0,0,\mathbf{p}_0\right) &= 0, \quad i=,1,2,\ldots r\end{aligned}$$

with

$$\Delta = \frac{\partial^{r+1}\mathcal{G}}{\partial x^{r+1}}\left(0,0,\mathbf{p}_0\right)\frac{\partial \mathcal{G}}{\partial \mu}\left(0,0,\mathbf{p}_0\right) \neq 0.$$

Then in the neighbourhood of $\mathbf{p}_0$*,* $\mathcal{G}\left(x,\mu\right)$ *is contact equivalent to*

$$\mathcal{F}\left(x,\mu\right) = x^{r+1} \mp \lambda \begin{cases} -sign & if\, \Delta < 0 \\ +sign & if\, \Delta > 0 \end{cases} \tag{2.7}$$

whose universal unfolding is

$$\mathcal{U}\left(x,\lambda,\alpha\right) = x^{r+1} - \alpha_{r-1}x^{r-1} - \alpha_{r-2}x^{r-2} - \ldots - \alpha_1 x \mp \lambda. \tag{2.8}$$

This theorem is a special case of a more general result proved in [27]. Why is this theorem useful? Because it shows that if such a singular point exists, then the maximum number of solutions of the equation

$$\mathcal{G}\left(x,\mu,\mathbf{p}\right) = 0$$

next to the singularity point $\mathbf{p}_0$ is $r+1$. Thus we can use singularity theory to find the maximum number of steady-state solutions.

We have already met the cases $r=1$ and $r=2$ which represent the limit-point bifurcation and the cusp singularity respectively. The maximum number of steady-state solutions exhibited by these singularities are two and three respectively. The cases $r=3$ and $r=4$ are known as the swallowtail and butterfly singularity respectively. They exhibit a maximum number of solutions of four and five respectively.

2.7.2 The double limit-point singularity revisited

There is another application of theorem 2.7 that is often very useful. In answering example 2.8 we realised that a polynomial of degree three can not exhibit the double limit-point singularity because the maximum number of steady-state solutions that a cubic can have is three: in the neighbourhood of the double limit-point singularity the function $\mathcal{G}$ must exhibit *four* steady-state solutions.

This is a very useful result, but it only applies to polynomials and most models contain other types of functions. It would be very useful to know for what kinds of singularity functions the double limit-point singularity can not occur. Theorem 2.7 provides the required characterisation. A double limit-point singularity can not occur if the maximum value of r is either one or two because in these cases the maximum number of steady-state solutions are two and three respectively.

Need to discuss quartic fold singularity here.

2.7.3 Some other higher-order singularities

The function $\mathcal{G}$ given in questions 1a, 1b & 3 are the 'universal unfoldings' representing the asymmetric cusp and the quartic fold respectively. These are higher-order singularities which we have not discussed.

2.8 Mathematical singularity theory: All that is gold does not glitter

Mathematical singularity theory enables singular points to be located in the secondary bifurcation parameter space $\mathbf{p}$. These points are characterised by the vanishing of the singularity function $\mathcal{G}$ and several of its partial derivatives. In the vicinity of such points qualitatively different steady-state diagrams exist. The theory predicts all the steady-state diagrams existing locally to a singular point [28].

For instance, a singular point satisfying

$$\mathcal{G} = \mathcal{G}_x \qquad\qquad = \mathcal{G}_{xx} = \mathcal{G}_{\mu x} = 0, \mathcal{G}_{xxx}\mathcal{G}_{\lambda\lambda}\ neq0,$$

is known as a *winged-cusp* and such a singular point has seven qualitatively different steady-state diagrams in a neighbourhood of $\mathbf{p}_0$ [28, pages 208 & 209].

In the early days of singularity theory it was claimed that theorem 2.7 was useful because it can be used to find all possible steady-state diagrams for a system which does not exhibit either the isola or double limit-point singularities. This is both true and false. It is true that it is ‘easy’ to find all possible steady-state diagrams of the universal unfolding. However, in practice you don’t just want to know how many different qualitative steady-state diagrams your system has. And neither do you just want to know what they look like. You actually want to know where in parameter space they occur!

Note that the original problem contained a set of secondary bifurcation parameters $\mathbf{p}$ whereas the universal unfolding contains a different set of secondary bifurcation parameters α. Knowledge about the location of the steady-state diagrams in the parameter space α will only be of interest if you can find a smooth and invertible change of coordinates, possible also including a scaling factor, that will convert your function $\mathcal{G}$ to the universal unfolding $\mathcal{U}$. In theory you can find the required the required transformation, in practice you can not.

To find the parameter regions corresponding to each type of steady-state diagram will need to apply the method described in section 2.5, i.e. you must construct the cusp, double-limit and isola singularities in the parameter space $\mathbf{p}$.

Another mistake in early singularity theory was to claim that identifying the highest order singularity allows you to identify the number of different possible steady-state diagrams in a model.

Need to extend this section

2.9 Conclusions

At the start of this chapter, in section 2.1, we posed three questions in the context of the spruce budworm model from chapter 1.6. These questions can be stated for a general model of the form

$$\mathcal{G}\left(x, \mu, \alpha, \beta\right) = 0 \tag{2.9}$$

as follows

1. What does it mean for two steady-state diagrams to differ in a qualitatively different manner?
2. Can we identify when a steady-state diagram changes between qualitatively different forms?
3. Is it possible to identify all qualitatively different steady-state diagrams *and* find their location in parameter space?

The mathematical basis for answering these questions is provided by singularity theory with a distinguished parameter. The answer to the first two questions is provided in section 2.4. Namely that a steady-state diagram can only change in a qualitative way if the secondary bifurcation parameters cross one of three surfaces. These surfaces represent the cusp singularity, the isola singularity and the double limit-point singularity. This provides the basis for answering the third question (section 2.5): parameterise the three surfaces in the space of the secondary bifurcation parameters.

Singularity theory, via theorem 2.7, can also be used to find the maximum number of steady-state solutions exhibited by equation 2.9.

An attractive feature of this method Balakotaiah and Luss) is that the location of the boundaries is determined directly in the physical parameter space, whereas in singularity theory proper, the boundaries are defined in terms of the unfolding parameters appearing in the normal form of the singularity; in practice it is very difficult to relate the unfolding parameters to the physical ones.

Singularity theory is thus an efficient and powerful tool, that is easy to apply in a systematic manner to specific models. It has become a standard tool, particularly in chemical engineering, for investigating multiplicity features of equations of the form of (2.9).

We have considered the special case in which the state variable (x) and the singularity function ($\mathcal{G}$) are scalar. Although the theory can handle it, we have not considered the more general case in which both of these quantities are vectors.

2.9.1 Background

Singularity theory grew out of an earlier mathematical theory called catastrophe theory. Catastrophe theory, which originated with work carried out by the French mathematician Reneè Thom in the 1960s [62], explains the circumstances in which a system which usually behaves smoothly can undergo a single abrupt change. It was popularised to the general public in the 1970s through the work of Christopher Zeeman [65]. Catastrophe theory was applied to problems from a wide range of disciplines, stretching from the physical sciences through the biological sciences and onto the social sciences. What we have called a limit-point bifurcation is often known as the “fold” in catastrophe theory.

A formal, rigorous survey of singularity theory is presented in [28].

This methodology of constructing a bifurcation diagram by calculating the locus of the cusp, isola and double-limit point singularities was first systematically applied to investigate multiplicity features of open chemically reacting systems by Balakotaiah and Luss [8, 9, 10, 11].

A heuristic description of this theory with a focus on applications to chemical systems has been written by Balakotaiah [7].

2.10 Maple commands

2.11 Revision of key ideas

The following questions are about the key ideas in this chapter.

2.12 Questions on singularity theory with a distinguished parameter

1. The singularity function for a single first-order exothermic reaction taking place in a cooled continuously stirred tank reactor is given by [10][2]

$$\mathcal{G}\left(Y,B,D\right) = Y - D\left(B - Y\right)\exp\left[Y\right]$$

   (a) Find the parameter values at which the cusp singularity occurs.

---

[2] To obtain the singularity equation the positive exponential approximation has been made. If you do not understand this, do not worry. It's of interest only to individuals working in combustion science.

(b) Can the swallowtail singularity occur?

(c) Can the isola singularity occur?

(d) Can the double limit-point singularity occur?

(e) Suppose that the primary bifurcation is B.

  (i) How many types of steady-state diagrams are there?

  (ii) Exhibit all types of steady-state diagram and specify the values of D over which they occur.

(f) Suppose that the primary bifurcation is D.

  (i) How many types of steady-state diagrams are there?

  (ii) Exhibit all types of steady-state diagram and specify the values of D over which they occur.

2. There is often more than one way to scale an equation and the scaling used should reflect a feature of interest. If we wish to investigate how the steady-state temperature for a single first-order exothermic reaction taking place in a cooled continuously stirred tank varies as a function of either the coolant or feed temperature then the governing equations should be scaled differently to that used to obtain the singularity function given in the previous question.

A more appropriate version of the singularity function for this problem is given by [10, page 1620][3]

$$\mathcal{G}\left(\theta,\sigma,Da,\beta\right) = \theta - \sigma - Da\left(\beta + \sigma - \theta\right)\exp\left[-\frac{1}{\theta}\right] = 0.$$

(a) Find the parameter values at which the cusp singularity occurs.

(b) Can the swallowtail singularity occur?

(c) Can the isola singularity occur?

(d) Can the double limit-point singularity occur?

(e) Suppose that the primary bifurcation is σ.

  (i) How many types of steady-state diagrams are there?

  (ii) Exhibit all types of steady-state diagram and specify the values of D over which they occur.

3. (This question is based upon work presented in a research paper and is likely to be non-trivial!)

The singularity function for two consecutive first-order exothermic reactions taking place in a cooled continuously stirred tank reactor is given by [10][4]

$$\mathcal{G}\left(\theta, Da_1, \nu, B_1, B_2\right) = Da_1\left(B_1 - \theta - \nu\theta\right)e^{\theta} + \nu\left(Da_1\right)^2\left(B_1 + B_2 - \theta\right)e^{2\theta} - \theta = 0.$$

Note that the state variable and all bifurcation parameters are non-negative.

(a) Find the unique physically meaningful parameter values at which the butterfly singularity occurs.

Note. There are two parameter sets at which this singularity occurs. However, one of them corresponds to a negative value for B_2 and is therefore not physically meaningful.

(b) Is it possible for this system to have a steady-state equation with six steady-state solutions?

(c) Can the isola singularity occur?

(d) It can be shown that there are four generic bifurcation diagrams in the (ν, B_1) plane depending upon the value for B_2. These regions are

  (i) $0 < B_2 < 6\sqrt{3}\left(2-\sqrt{3}\right)$.

  (ii) $6\sqrt{3}\left(2-\sqrt{3}\right) < B_2 < \frac{27}{8}$

  (iii) $\frac{27}{8} < B_2 < 4$

---

[3] To obtain the singularity equation the positive exponential approximation has been made. If you do not understand this, do not worry. It's of interest only to individuals working in combustion science.

[4] To obtain the singularity equation it has been assumed that the activation energies of the two reactions are identical and that the intermediate species is not present in the feed. Furthermore, the positive exponential approximation has been made. If you do not understand this, do not worry. It's of interest only to individuals working in combustion science.

(iv) $4 < B_2$

For each region pick a value for B_2 and determine the corresponding bifurcation diagram.

Hint. First draw the cusp locus and determine how many regions the plane is divided into. For each region find the number of limit points on the steady-state diagram. This will tell you if the double-limit point singularity can occur in that region.

Hint 2. The first two bifurcation diagrams are much easier than the last two. You shouldn't really bother attempting the last two... unless you are doing this question as part of a project!

2.13 To do

1. The application of singularity theory may reduce the amount of systematic experimentation required to establish the system's properties.
2. Explain the significance of $G_{xxx} = 0$ for the double-limit point singularity. When this has been done rewrite the end of section 2.4.3.
3. Look at questions for assignment to round out the discussion.
4. Better answer question 1.19.
5. Need to discuss quartic fold singularity in section 2.7.2.
6. Boundary bifurcation. Maybe in an extra chapter at the end of the course on advanced singularity theory. A good example is a single zeroth-order reaction in a CSTR [10, pages 1615 & 1616]
7. Further examples: Balakotaiah and Luss 1983. Chem Eng Sci 38, 1709-1721.
   Balakotaiah and Luss 1982. Chem Eng Sci 39, 865-881.
8. Balakotaiah and Luss 1982. Chem Eng Sci 37, 433 has more detail on the multiplicity for two consecutive exothermic reactions.
9. Mathematical theory for the butterfly catastrophe are reported in detail in: Brocker, Th. and Lander, L., *Differentiable Germs and Catastrophe.* Cambridge University Press (1975).
   Poston, T., and Stewart, I. *Catastrophe THeory and its Applications.* Pitman. London (1978).
10. Probably should read [27] at some stage.

Need to add this in an appropriate place

2.13.1 Unfolding diagram

It is sometimes useful to construct a figure showing how the value(s) of the primary bifurcation parameter at a particular type of bifurcation point vary as a secondary bifurcation parameter is changed. Such a figure is sometimes called an *unfolding diagram.*

For the singularity function

$$\mathcal{G} = x^3 - \mu + \alpha x$$

we showed in example 2.4 that a cusp singularity occurs when $\alpha = 0$. For this particular model there are two limit points when $\alpha < 0$ and no limit points when $\alpha > 0$. Thus for a given value of the secondary bifurcation parameter, provided that $\alpha = \alpha_0 < 0$, there are two values of the primary bifurcation parameter (μ), μ_1 and μ_2, corresponding to a limit-point bifurcation on the steady-state diagram.

Question 2.8 *Construct an unfolding diagram for the limit points that occur in the singularity function*

$$\mathcal{G} = x^3 - \mu + \alpha x.$$

On your diagram identify the location of the cusp point.

To do. Maybe I should even give the figure in the notes, to clarify what the figure should look like. Note hard to show that $\alpha = -3x^2$ and $\mu = \left(\frac{-\alpha}{3}\right)^{3/2} + \alpha\left(\frac{-\alpha}{3}\right)^{1/2}$.

Maybe better is $\alpha = -3x^2$ and $\mu = -2x^3$.

2.13.2 Notation

There is no consistency of notation in the literature. What we call a steady-state diagram has been referred to as a bifurcation diagram by some authors. Similarly what we call an unfolding diagram has also been called a bifurcation diagram.

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