Chapter 7

Second-order differential equations: Bifurcations and steady-state diagrams

7.1 Introduction

In this chapter we consider the planar system

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= f\left(x, y, \lambda, \alpha, \beta\right), \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= g\left(x, y, \lambda, \alpha, \beta\right).\end{aligned} \tag{7.1}$$

In chapter 3 we learnt how to find the steady-state solutions of system 7.1 and how to determine their stability. In chapter 6 we learnt that stability can be changed at a Hopf bifurcation and that Hopf bifurcations are associated with the birth/death of limit cycles. Thus a steady-state diagram for system (7.1) may contains two types of information. Firstly, it shows how the steady-state values for the state variables (x and y) vary as a function of the primary bifurcation parameter (λ). This can be thought of as 'static' information because these solutions are independent of time. Secondly, the steady-state diagram shows information connected with limit-cycles (should they exist). This information might be the average value of the state variables along one period of the limit cycle or the minimum/maximum values of the state variables along one period of the limit cycle. This can be thought of as 'dynamic' information because a limit cycle is a function of time.

We should expect, based upon chapter 2, that as the values for the secondary bifurcation parameters (α and β) change that the 'static information' changes in a 'fundamental' way. Thus, the number of steady-state solutions as a function of the primary bifurcation parameter may change, as may the 'shape' of the steady-state diagram. This is known as 'static multiplicity'. We might also anticipate that the 'dynamic' information on a steady-state diagram changes as the second bifurcation parameters are varied. Thus both the number of Hopf bifurcation points on the steady-state diagram and the number of limit cycles as a function of the primary bifurcation might change. This is known as 'dynamic multiplicity'.

In this chapter we investigate questions associated with static multiplicity, such as how many steady-state diagrams does system 7.1 have as the secondary bifurcation parameters are varied?' We will investigate questions associated with 'dynamic multiplicity' in chapter 8.

We first note that if we can reduce the analysis of system 7.1 down to a single equation

$$\mathcal{G}\left(x, \lambda, \alpha, \beta\right) = 0,$$

then our problem reduces to that studied in chapter 2. Such a reduction is possible if we can solve one of the steady-state equations

$$\begin{aligned}f\left(x, y, \lambda, \alpha, \beta\right) &= 0, \\ g\left(x, y, \lambda, \alpha, \beta\right) &= 0,\end{aligned}$$

for one of the state variables (x or y). Suppose that we can solve the first equation for x, so that

$$x = h\left(y, \lambda, \alpha, \beta\right).$$

Substituting this into the second equation gives

$$g\left(h\left(y, \lambda, \alpha, \beta\right), y, \lambda, \alpha, \beta\right) = \mathcal{G}\left(y, \lambda, \alpha, \beta\right) = 0.$$

7.2 Singularity theory and bifurcation points

In chapter 2.3 we learnt how to use singularity theory to find the three types of static bifurcations (limit point bifurcation, transcritical bifurcation and pitchfork bifurcation) for a scalar equation. In this section we give defining conditions for these bifurcations for system 7.1.

7.2.1 The limit point theorem

Theorem 7.1 (Limit point bifurcation) *Suppose that at the point* $(\lambda, x, y) = (\lambda_0, x_0, y_0)$ *the following equations are satisfied*

$$\begin{aligned} f = g = \frac{d\lambda}{dx} &= 0, \\ \frac{d^2\lambda}{dx^2} &\neq 0, \\ \frac{d^2\lambda}{dx^2} &\neq \infty. \end{aligned}$$

Then a limit point bifurcation occurs at the point (λ_0, x_0, y_0).

In calculating $\dfrac{\mathrm{d}\lambda}{\mathrm{d}x}$ we regard y and λ as implicit functions of x.

Example 7.1 ([30], pages 371–372.)

1. *Find the conditions necessary for a limit-point bifurcation for the system*

$$\begin{aligned} \frac{dx}{dt} &= f\left(x, y, \lambda, \alpha, \beta\right) &&= x^2 + \lambda y + \beta, \\ \frac{dy}{dt} &= g\left(x, y, \lambda, \alpha, \beta\right) &&= \alpha x + \lambda^2 - y. \end{aligned} \tag{7.2}$$

2. *Obtain an expression for the limit-point curve in the* $\lambda - \alpha$ *plane.*

Solution

1. The steady-state solutions are defined by $f = g = 0$ and are given by

$$\begin{aligned} x^2 + \lambda y + \beta &= 0, \\ \alpha x + \lambda^2 - y &= 0. \end{aligned}$$

Regarding y and λ as implicit functions of x and differentiating both expressions with respect to x we obtain

$$\begin{aligned} 2x + \lambda\frac{\mathrm{d}y}{\mathrm{d}x} + y\frac{\mathrm{d}\lambda}{\mathrm{d}x} &= 0, \\ \alpha + 2\lambda\frac{\mathrm{d}\lambda}{\mathrm{d}x} - \frac{\mathrm{d}y}{\mathrm{d}x} &= 0. \end{aligned}$$

These equations are a pair of simultaneous linear equations for the derivatives $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}\lambda}{\mathrm{d}x}$. Solving we obtain

$$\begin{aligned}\frac{\mathrm{d}\lambda}{\mathrm{d}x} &= -\frac{(2x+\lambda\alpha)}{y+2\lambda^2},\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \alpha - \frac{2\,(2x+\alpha\lambda)\,\lambda}{2\lambda^2+y}.\end{aligned}$$

The defining conditions for a limit-point bifurcation in system (7.2) are therefore

$$\begin{aligned}x^2+\lambda y+\beta &= 0,\\ \alpha x+\lambda^2-y &= 0,\\ 2x+\lambda\alpha &= 0.\end{aligned}$$

2. After some algebra we obtain the equation

$$4\left(\lambda^3+\beta\right) = \alpha^2\lambda^2$$

from which we deduce that

$$\alpha = \begin{cases} \frac{\pm\lambda}{2}\left(\lambda^3+\beta\right)^{1/2} & \beta \neq 0\\ \pm 2\sqrt{\lambda} \quad \text{and} \quad \lambda = 0 & \beta = 0.\end{cases}$$

□

Question 7.1 *Show the locus of limit points in the* $\lambda-\alpha$ *plane for system* (7.2). *Consider the cases* $\beta > 0$, $\beta = 0$ *and* $\beta < 0$. *(These are shown in [30, figure 9].)*

7.2.2 The transcritical bifurcation

Theorem 7.2 (Transcritical bifurcation) *Suppose that at the point* $(\lambda, x, y, \alpha) = (\lambda_0, x_0, y_0, \alpha_0)$ *the following equations are satisfied*

$$\begin{aligned}&f = g = \frac{d\lambda}{dx} = \frac{d\alpha}{d\lambda} = 0,\\ &\frac{d^2\alpha}{d\lambda^2} \neq 0,\\ &\frac{d^2\lambda}{dx^2} \neq 0.\end{aligned}$$

Then a transcritical bifurcation occurs at the point (λ_0, x_0, y_0).

7.2.3 The pitchfork bifurcation

Theorem 7.3 (Pitchfork bifurcation) *Suppose that at the point* $(\lambda, x, y) = (\lambda_0, x_0, y_0, \alpha_0, \beta_0)$ *the following equations are satisfied*

$$\begin{aligned}&f = g = \frac{d\lambda}{dx} = \frac{d^2\lambda}{dx^2} = \frac{d\alpha}{d\lambda} = 0,\\ &\frac{d^2\alpha}{d\lambda^2} \neq 0,\\ &\frac{d^3\lambda}{dx^3} \neq 0.\end{aligned}$$

Then a pitchfork bifurcation occurs at the point (λ_0, x_0, y_0).

7.3 Steady-state diagrams

In chapter 2.4 we learnt that that there are three mechanisms by which a steady-state diagram can change in a qualitative manner. These are the:

1. the *cusp* (hysteresis) singularity;
2. the *isola* singularity;
3. the *double limit-point* singularity.

In this section we give the defining conditions for these singularities for system 7.1.

7.3.1 The cusp singularity

Theorem 7.4 (Cusp singularity) *Suppose that at the point* $(\lambda, x, y) = (\lambda_0, x_0, y_0, \alpha_0)$ *the following equations are satisfied*

$$\begin{aligned} f = g = \frac{d\lambda}{dx} = \frac{d^2\lambda}{dx^2} &= 0, \\ \frac{d^3\lambda}{dx^3} &\neq 0, \\ \frac{d\alpha}{dx} &\neq \infty. \end{aligned} \tag{7.3}$$

Then a cusp singularity occurs at the point $(\lambda_0, x_0, y_0, \alpha_0)$.

In calculating $\frac{\mathrm{d}\lambda}{\mathrm{d}x}$ we regard y and λ as implicit functions of x.

Example 7.2 ([30], pages 371–372.)

1. *Find the conditions necessary for a cusp singularity for the system*

$$\begin{aligned} \frac{dx}{dt} &= f(x, y, \lambda, \alpha, \beta) &&= x^2 + \lambda y + \beta, \\ \frac{dy}{dt} &= g(x, y, \lambda, \alpha, \beta) &&= \alpha x + \lambda^2 - y. \end{aligned} \tag{7.4}$$

2. *Where does the cusp singularity occur in the* $(\alpha - \beta)$ *plane?*

Solution

1. From our solution to question 7.1 we know that a limit point occurs when.

$$\begin{aligned} x^2 + \lambda y + \beta &= 0, \\ \alpha x + \lambda^2 - y &= 0, \\ 2x + \lambda\alpha &= 0. \end{aligned}$$

We further know that

$$\frac{\mathrm{d}\lambda}{\mathrm{d}x} = -\frac{(2x + \lambda\alpha)}{y + 2\lambda^2}.$$

Implicit differentiation of (7.4) shows that

$$\frac{\mathrm{d}^2\lambda}{\mathrm{d}x^2} = \frac{-2\lambda}{\lambda^3 - 2\beta}.$$

2. A cusp singularity can not occur. Not that $\dfrac{\mathrm{d}^2\lambda}{\mathrm{d}x^2} = 0 \Rightarrow \lambda = 0$ and $\beta \neq 0$.

   Then $\dfrac{\mathrm{d}\lambda}{\mathrm{d}x} = 0 \Rightarrow x = 0$ and $y \neq 0$. But then there is no steady-state value for y because the equation

   $$\alpha x + \lambda^2 - y = 0$$

   forces $y = 0$ when $\lambda = x = 0$.

□

7.3.2 The isola singularity

Theorem 7.5 (Isola singularity) *Suppose that at the point* $(\lambda, x, y, \alpha) = (\lambda_0, x_0, y_0, \alpha_0)$ *the following equations are satisfied*

$$\begin{aligned} f = g = \frac{d\lambda}{dx} = \frac{d\alpha}{d\lambda} &= 0, \\ \frac{d^2\alpha}{d\lambda^2} &\neq 0, \\ \frac{d^2\lambda}{dx^2} &\neq 0. \end{aligned}$$

Then an isola singularity occurs at the point $(\lambda_0, x_0, y_0, \alpha_0)$.

7.3.3 The double limit point singularity

Theorem 7.6 (Double limit point singularity) *Suppose that at the points* $(\lambda, x, y, \alpha) = (\lambda_0, x_0, y_0, \alpha_0)$ *and* $(\lambda, x, y, \alpha) = (\lambda_0, x_1, y_1, \alpha_0)$ *the following equations are satisfied*

$$\begin{aligned} f(\lambda_0, x_0, y_0, \alpha_0) &= f(\lambda_0, x_1, y_1, \alpha_0) = 0, \\ g(\lambda_0, x_0, y_0, \alpha_0) &= g(\lambda_0, x_1, y_1, \alpha_0) = 0, \\ \frac{d\lambda}{dx}(\lambda_0, x_0, y_0, \alpha_0) &= \frac{d\lambda}{dx}(\lambda_0, x_1, y_1, \alpha_0) = 0, \\ \frac{d^2\lambda}{dx^2}(\lambda_0, x_0, y_0, \alpha_0) \neq 0 &\quad \frac{d^2\lambda}{dx^2}(\lambda_0, x_1, y_1, \alpha_0) \neq 0, \\ \frac{d^2\lambda}{dx^2}(\lambda_0, x_0, y_0, \alpha_0) \neq \infty &\quad \frac{d^2\lambda}{dx^2}(\lambda_0, x_1, y_1, \alpha_0) \neq \infty. \end{aligned}$$

Then a double limit point bifurcation occurs at the points $(\lambda, x, y, \alpha) = (\lambda_0, x_0, y_0, \alpha_0)$ *and* $(\lambda, x, y, \alpha) = (\lambda_0, x_1, y_1, \alpha_0)$.

7.4 Conclusions

7.4.1 Summary

7.4.2 Further reading

7.5 Maple commands

7.6 Revision of key ideas

The following questions are about the key ideas in this chapter.

7.7 Questions on bifurcations and steady-state diagrams

1. A question.

7.8 Things to do

1. Figure 2 from [30]
2. Explain how to use the `implicitdiff` command.