Chapter 4

Second-order differential equations: The absence of periodic solutions

4.1 Introduction

Consider the differential equation

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= f(x,y), \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= g(x,y). \end{aligned} \tag{4.1}$$

A *periodic solution* of this system is a solution ϕ that is not a steady-state of the system and which has the property that

$$\phi(t+T) = \phi(t),$$

for some $T > 0$. The minimal T for which this equality holds is called the *period* of the periodic orbit ϕ. In the phase plane a periodic solution is represented by a closed path.

An *isolated* periodic solution, $\mathcal{C}$, of an autonomous system is said to be a *limit cycle*. By an isolated periodic solution, we mean that there are no other periodic solutions in an arbitrarily small neighbourhood of $\mathcal{C}$. Solutions that start sufficiently near to an isolated periodic solution must either spiral into or away from the limit cycle. The existence of limit cycles is a feature of great practical importance.

A 'limit cycle' is an 'isolated periodic solution'? Wouldn't all periodic solutions have to be isolated? The following example, which we used in chapter 3 as an example of a neutrally stable steady-state, shows that this is not the case.

Example 4.1 (A system with periodic solutions, none of which are isolated) *Consider the system*

$$\begin{aligned} \dot{x} &= -y^5, \quad x(0) = x_0, \\ \dot{y} &= x^5, \quad\ \ y(0) = y_0. \end{aligned}$$

We know from question 3.7 that this system has an infinite number of periodic solutions that encircle the origin. These solutions satisfy

$$x^6 + y^6 = x_0^6 + y_0^6.$$

Consider the periodic solution associated with a specified initial condition. Change the initial condition by an arbitrarily small amount. The new initial condition specifies a new periodic solution. (In fact there are an infinite number of periodic solutions between those specified by the original and the new initial conditions). Thus the periodic solution specified by the original initial condition is *not* isolated. □

In this chapter we provide conditions under which a system of two first order differential equations does not have any periodic solutions (and consequently no limit cycles).

4.2 Bendixson's criteria and Dulac's test

We first describe Bendixson's criteria, theorem 4.1, which can be used to show that periodic solutions are not possible in a simply connected region $\mathcal{D}$[1]. We then describe Dulac's test, theorem 4.2, which is a generalisation of Bendixson's criteria.

Theorem 4.1 (Bendixson's Criteria [18]) *Consider the system*

$$\frac{dx}{dt} = f(x,y),$$
$$\frac{dy}{dt} = g(x,y).$$

Let $\mathcal{D}$ be a simply connected region in $\mathbb{R}^2$ and let $\{f(x,y), g(x,y)\} \in C^1(\mathcal{D})$. If the divergence of the vector field $\mathbf{f}$,

$$\nabla \cdot \mathbf{f} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y},$$

is not identically zero and does not change sign in $\mathcal{D}$, then this system does not have any closed orbits lying entirely in $\mathcal{D}$.

Proof Suppose that $\Gamma : \mathbf{x} = \mathbf{x}(t), 0 \leq t \leq T$, is a close orbit of the system lying entirely in $\mathcal{D}$. If $\mathcal{S}$ denotes the interior of Γ, it follows from Green's Theorem for integrals in the plane that

$$\begin{aligned}\iint_{\mathcal{S}} \nabla \cdot \mathbf{f}\, \mathrm{d}x\mathrm{d}y &= \oint_{\Gamma} \left(f(x,y)\ \mathrm{d}y - g(x,y)\ \mathrm{d}x\right), \\ &= \int_0^T \left(f(x,y)\frac{\mathrm{d}y}{\mathrm{d}t} - g(x,y)\frac{\mathrm{d}x}{\mathrm{d}t}\right)\mathrm{d}t, \\ &= \int_0^T \left(f(x,y)\,g(x,y) - g(x,y)\,f(x,y)\right)\mathrm{d}t,\end{aligned}$$

as on the orbit Γ (4.1) holds

$$= 0.$$

And if $\nabla \cdot \mathbf{f}$ is not identically zero and does not change sign in $\mathcal{S}$, then it follows from the continuity of $\nabla \cdot \mathbf{f}$ in $\mathcal{S}$ that the above double integral is either positive or negative. In either case this leads to a contradiction. Therefore, there is no closed orbit of the system lying entirely in $\mathcal{D}$. □

Note that no conclusions can be drawn for systems where Bendixson's criteria fails. Such systems may, or may not, contain a periodic solution.

A more general result, which is also proved by contradiction using Green's theorem in the plane, is Dulac's test.

Theorem 4.2 (Dulac's Test [29]) *Consider the system*

$$\frac{dx}{dt} = f(x,y),$$
$$\frac{dy}{dt} = g(x,y).$$

Let $\mathcal{D}$ be a simply connected region in $\mathbb{R}^2$ and let $\{f(x,y), g(x,y)\} \in C^1(\mathcal{D})$. If there exists a function $\rho \in C^1(\mathcal{D})$ such that

$$\frac{\partial \rho f}{\partial x} + \frac{\partial \rho g}{\partial y}$$

is not identically zero and does not change sign in $\mathcal{D}$, then this system does not have any closed paths lying entirely in $\mathcal{D}$.

Bendixson's criteria, theorem 4.1, is simply Dulac's test, 4.2, with the choice $\rho = 1$.

[1] A simply connected region does not contain any holes or disjoint regions.

4.3 Applying Bendixson's criteria and Dulac's test

Question 4.1 *Given a system of two differential equations should we first apply Bendixson's criteria or Dulac's test?*

It does not matter if you want to call it Bendixson's criteria or Dulac's test with the choice $\rho = 1$, but given a system you should always calculate

$$\nabla \cdot \mathbf{f} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}.$$

If you are lucky you may be able to show that periodic solutions can not occur! □

Example 4.2 ([44], exercise 22 on page 91) *Show that the system*

$$\dot{x} = y + x^3,$$
$$\dot{y} = x + y + y^3.$$

has no periodic solutions.

Take $\rho(x, y) = 1$ and apply Dulac's test.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left[\rho f(x,y)\right] + \frac{\mathrm{d}}{\mathrm{d}y}\left[\rho g(x,y)\right] &= \frac{\mathrm{d}}{\mathrm{d}x}\left[y + x^3\right] + \frac{\mathrm{d}}{\mathrm{d}y}\left[x + y + y^3\right] \\ &= 3x^2 + 1 + 3y^2,\end{aligned}$$

which is always positive. Hence by Dulac's test with $\rho = 1$ (or Bendixson's criteria) the system has no periodic solutions. □

Sometimes we can rule out the possibility of periodic solutions under some, but not all, circumstances.

Example 4.3 *Consider the system*

$$\dot{x} = y,$$
$$\dot{y} = -\left(\mu + x^2 + x^4\right) y - x.$$

What can you deduce about the possible existence of periodic solutions in this system?

Take $\rho(x, y) = 1$ and apply Dulac's test.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left[\rho f(x,y)\right] + \frac{\mathrm{d}}{\mathrm{d}y}\left[\rho g(x,y)\right] &= \frac{\mathrm{d}}{\mathrm{d}x}\left[y\right] + \frac{\mathrm{d}}{\mathrm{d}y}\left[-\left(\mu + x^2 + x^4\right) y - x\right] \\ &= -\left(\mu + x^2 + x^4\right).\end{aligned}$$

This function is strictly negative for $\mu > 0$ and never positive for $\mu = 0$. Hence by Dulac's test with $\rho = 1$ (or Bendixson's criteria) the system has no periodic solutions for $\mu \geq 0$. □

Sometimes when we can not rule out the possibility of a periodic solution we can say something about where in phase-space any possible periodic solution can *not* be found.

Example 4.4 *Consider the system*

$$\dot{x} = y,$$
$$\dot{y} = -\left(\mu + x^2 + x^4\right) y - x.$$

What can you deduce about the possible existence of periodic solutions in this system with the restriction that $\mu < 0$*?*

We know from the previous example that taking $\rho = 1$ in Dulac's test leads to the conclusion that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\rho f(x,y)\right] + \frac{\mathrm{d}}{\mathrm{d}y}\left[\rho g(x,y)\right] = -\left(\mu + x^2 + x^4\right).$$

Consider the function

$$\mathcal{F} = x^4 + x^2 + \mu.$$

This function has the property that

$$\mathcal{F} = \begin{cases} > 0, & |x| > x_{\mathrm{cr}}, \\ = 0, & |x| = x_{\mathrm{cr}}, \\ < 0, & |x| < x_{\mathrm{cr}}, \end{cases}$$

where

$$x_{\mathrm{cr}} = \left[\frac{-1+\sqrt{1-4\mu}}{2}\right]^{0.5}.$$

Thus there are no periodic solutions whose phase path lies entirely in the region $x \le -x_{\mathrm{cr}}$, or entirely in the region $-x_{\mathrm{cr}} \le x \le x_{\mathrm{cr}}$, or entirely in the region $x_{\mathrm{cr}} \le x$. □

Sometimes it is possible to prove a result about a general class of equations.

Example 4.5 ([44], page 82.) *Show that the equation*

$$\frac{d^2 x}{dt^2} + f(x)\frac{dx}{dt} + g(x) = 0$$

can have no periodic solution whose phase path lies in a region where f is of one sign.

The equivalent system is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y,$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = -f(x)\,y - g(x)$$

$$\text{Take} \quad \rho(x,y) = 1$$

and apply Dulac's test.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left[\rho f(x,y)\right] + \frac{\mathrm{d}}{\mathrm{d}y}\left[\rho g(x,y)\right] &= \frac{\mathrm{d}}{\mathrm{d}x}\left[1\cdot f(x,y)\right] + \frac{\mathrm{d}}{\mathrm{d}y}\left[1\cdot g(x,y)\right] \\ &= -f(x),\end{aligned}$$

which is of one sign whenever f is. Hence by Dulac's test with $\rho = 1$ (or Bendixson's criteria) the system has no periodic solutions for $\mu \ge 0$. □

Question 4.2 *What happens if Dulac's test with $\rho = 1$ provides either no information or only incomplete information, such as in example 4.3?*

If we are sufficiently clever, or we have sufficient patience, we might be able to find a function $\rho(x,y)$ which shows that our system never has a periodic solution. Some of the examples from sections 4.7.2 & 4.7.3 show what can 'sometimes' be achieved with appropriate ingenuity. Only 'sometimes'? Yes, because for some parameter values your system might have periodic solutions! □

It might be futile to try and prove that there are circumstances under which periodic solutions can not occur. Can we instead show that there are circumstances under which limit cycles must occur? This is the topic for chapter 6.

4.4 Conclusions

In this chapter we have provided a method, Dulac's test, which can be used to show that the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x,y),$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = g(x,y),$$

does not have a periodic solution in a simply connected region $\mathcal{D}$. To do this we must find a function $\rho(x,y)$, possible $\rho(x,y) \equiv 1$, which has the property that

$$\nabla \cdot (\rho \mathbf{f}) = \frac{\partial \rho f}{\partial x} + \frac{\partial \rho g}{\partial y}$$

is not identically zero and does not change sign in $\mathcal{D}$. If this is true then the system does not have any closed orbits lying entirely in $\mathcal{D}$.

Sometimes this method will give powerful results with the choice of $\rho(x,y) \equiv 1$, or with another 'simple' function. A disadvantage of this method is that you may not be able to find an appropriate function function $\rho(x,y)$ in a 'reasonable' amount of time. Furthermore, you may be searching in fine for a function that does not exist!

4.5 Maple commands

For system (4.1) it is straightforward to write maple code to calculate the function

$$\nabla \cdot (\rho \mathbf{f}) = \frac{\partial \rho f}{\partial x} + \frac{\partial \rho g}{\partial y}.$$

However, it may not be straightforward to find a function ρ that has the required properties.

```
# dulac.txt
# 13.04.10
#
# Example maple code to apply Dulac's test.

eqn1 := y;                          # dx/dt
eqn2 := -(mu +x^2 +x^4)*y -x;       # dy/dt

rho := 1;                           # The dulac function rho

# We need to show that the function fn is not identicaly
# zero (Maple should show you if this is true) and does not
# change sign in the region D. The latter may not be easy!

fn := diff(rho*eqn1,x) + diff(rho*eqn2,y);

eqn1 := 'eqn1':
eqn2 := 'eqn2':
fn   := 'fn':
rho  := 'rho':
```

4.6 Revision of key ideas

The following questions are about the key ideas in this chapter.

1. What is meant by a *periodic solution* of a system of ordinary differential equations?
2. What is meant by a *limit cycle* of a system of ordinary differential equations?
3. Give an example of a system of two first-order differential equations that has periodic solutions that are not limit cycles.
4. Explain how to use Bendixson's criteria to show that a system of two first-order differential equations does not have any periodic solutions lying entirely in a simply connected region $\mathcal{D}$.
5. Proof Bendixson's criteria.
6. Explain how to use Dulac's test to show that a system of two first-order differential equations does not have any periodic solutions lying entirely in a simply connected region $\mathcal{D}$.

4.7 Questions on Bendixson's criteria and Dulac's test

4.7.1 Bendixson's criteria

1. (a) The equation

$$u'' + \phi(u)\, u' + \psi(u) = 0,$$

where the function $\phi(u)$ is always positive for all u (or is always negative for all u, describes a nonlinear oscillator with positive (or negative) damping for all u. Show that this system has no periodic solutions. [38, pages 144–145]

(This was an exam question in 2009).

(b) Consider the system

$$\begin{aligned}\dot{x} &= \mu x - \omega y + (\alpha x - \beta y)\left(x^2 + y^2\right),\\ \dot{y} &= \omega x - \mu y + (\beta x + \alpha y)\left(x^2 + y^2\right),\end{aligned}$$

where μ, ω, α and β are constants. Under what parameter restrictions are periodic solutions impossible? [55, example 3.3 on pages 152–153]

(c) Consider the system

$$\begin{aligned}\dot{x_1} &= x_2,\\ \dot{x_2} &= -x_1 + x_1^3 - 2\mu x_2.\end{aligned}$$

Under what parameter restrictions are periodic solutions impossible? [55, example 3.4 on page 153]

(d) Consider the system

$$\begin{aligned}\dot{x_1} &= x_2,\\ \dot{x_2} &= -\omega_0^2\left(x_1 - 1.402x_1^3 + 0.271x_1^5\right) - \left(2\mu_1 x_2 + \mu_3 x_2^3\right),\end{aligned}$$

where $\omega_0 = 5.278$. Under what parameter restrictions are periodic solutions impossible? [55, example 3.5 on page 153]

(e) Show that the unforced Duffing oscillator

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= x - x^3 - \delta y,\\ \delta &\geq 0\end{aligned}$$

has no limit cycles for $\delta > 0$. [79, pages 26–27]

(f) Consider the system

$$x' = x\left[\mu(Q) - 1\right],$$
$$Q' = \rho(1 - Qx, Q) - Q\mu(Q),$$

which is defined in the region

$$\mathcal{L} = \left\{(x, Q) \in \mathbb{R}^2_+ : xQ \leq 1, Q \geq P\right\}.$$

The function $\mu(Q)$ has the following properties for $Q \geq P$

$$\begin{aligned} \mu(Q) &\geq 0, \\ \mu'(Q) &> 0 \quad \text{and continuous for } Q \geq P \\ \mu(P) &= 0. \end{aligned}$$

The function $\rho(S, Q)$ is continuously differentiable in (S, Q) for $S \geq 0$ and $Q \geq P$ and has the following properties

$$\begin{aligned} \rho(0, Q) &= 0, \\ \rho(S, Q) &> 0 \quad \text{when } S > 0, \\ \frac{\partial \rho}{\partial S} &> 0, \\ \frac{\partial \rho}{\partial Q} &\leq 0. \end{aligned}$$

This is a simplification of a model for the growth of a population in a bioreactor feeding on a substrate. In this model x corresponds to the number of cells (or a concentration) and Q is the 'cell quota', which is the average amount of stored nutrient per cell at time t. The product xQ is the total "biomass" in the vessel. The function $\mu(Q)$ is a growth rate expression. The function $\rho(S, Q)$ models the uptake of free nutrient.

Can limit cycles occur in this system? [72, page 188]

2. It can be shown that the system

$$\dot{x} = x - y - x^3,$$
$$\dot{y} = x + y - y^3,$$

has a limit cycle in the annular region $A = \left\{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{x}| < \sqrt{2}\right\}$.

Show that the divergence $\nabla \cdot \mathbf{f} < 0$ in the region A. Why doesn't this contradict Bendixson's criteria? [62, problem 9.3 (a) on page 263]

4.7.2 Applications of Dulac's test

1. Prove Dulac's test. Hint, use the proof of Bendixson's criteria.

2. (a) By using Dulac's test with $\rho = e^{-2x}$, show that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y,$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = -x - y + x^2 + y^2,$$

has no limit cycles. [44, exercise 28 on page 92]

(b) Consider the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = af'(y),$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = g(x) - h(y) + f(y).$$

What restrictions (if any) are required on the functions $g(x)$ and $h(y)$ and a constant b for this system to pass Dulac's test with $\rho = e^{bx}$? Hoare & Nelson (November 2004)

(c) Use the Dulac function $\rho(x, y) = be^{-2\beta x}$ to show that the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = -ax - by + \alpha x^2 + \beta y^2$$

has no limit cycle in $\mathbb{R}^2$. [62, problem 9.5 (a) on page 264]

3. By using Dulac's test with $\rho = \dfrac{1}{xy}$ show that a population model for two interacting species of the form

$$x' = xf(x, y),$$
$$y' = yg(x, y),$$

has no periodic solution in the interior of the first quadrant when the functions f and g have the following properties for $x > 0$ and $y > 0$: $f_x < 0$, $g_y < 0$. [24, theorem 4.9 on pages 159–160]

4. By using Dulac's test show that the system

$$\frac{\mathrm{d}A}{\mathrm{d}\tau} = A + \frac{2\beta}{3\alpha}L - A^2,$$
$$\frac{\mathrm{d}L}{\mathrm{d}\tau} = A + \frac{\beta}{3\alpha} - AL,$$

where the coefficients α and β are positive, has no limit cycles in the open first quadrant. [72, page 223]

These equations come from a model describing the competition between two size-structured populations for a single nutrient.

5. Use Dulac's test to show that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x(y - 1),$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = x + y - 2y^2,$$

has no limit cycles. [Jordan & Smith]

6. Show that the following systems have no limit cycles:

$$\begin{array}{lll}
\text{(i)} & \dfrac{\mathrm{d}x}{\mathrm{d}t} = y & \dfrac{\mathrm{d}y}{\mathrm{d}t} = 1 + x^2 - (1-x)\,y, \\
\text{(ii)} & \dfrac{\mathrm{d}x}{\mathrm{d}t} = -(1-x)^3 + xy^2 & \dfrac{\mathrm{d}y}{\mathrm{d}t} = y + y^3 \\
\text{(iii)} & \dfrac{\mathrm{d}x}{\mathrm{d}t} = 2xy + x^3, & \dfrac{\mathrm{d}y}{\mathrm{d}t} = -x^2 + y - y^2 + y^3, \\
\text{(iv)} & \dfrac{\mathrm{d}x}{\mathrm{d}t} = x & \dfrac{\mathrm{d}y}{\mathrm{d}t} = 1 + x + y^2, \\
\text{(v)} & \dfrac{\mathrm{d}x}{\mathrm{d}t} = y & \dfrac{\mathrm{d}y}{\mathrm{d}t} = -1 - x^2, \\
\text{(vi)} & \dfrac{\mathrm{d}x}{\mathrm{d}t} = 1 - x^3 + y^2 & \dfrac{\mathrm{d}y}{\mathrm{d}t} = 2xy; \\
\text{(vii)} & \dfrac{\mathrm{d}x}{\mathrm{d}t} = y & \dfrac{\mathrm{d}y}{\mathrm{d}t} = \left(1 + x^2\right) y + x^3 \\
\text{(viii)} & \dfrac{\mathrm{d}x}{\mathrm{d}t} = x + y + x^3 - y^3 & \dfrac{\mathrm{d}y}{\mathrm{d}t} = -x + 2y - x^2 y + y^3 \\
\text{(ix)} & \dfrac{\mathrm{d}x}{\mathrm{d}t} = -xe^y + y^2 & \dfrac{\mathrm{d}y}{\mathrm{d}t} = x - x^2 y \\
\text{(x)} & \dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{y}{1+x^2} & \dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{-x + y\left(1 + x^2 + x^4\right)}{1+x^2}
\end{array}$$

Parts (i–vii) from [44, exercise 29 on npage 92]. Parts (viii–ix) from [38, problem 2 on page 169]. Part (x) from [62, problem 9.5 (b) on page 264].

7. (a) Consider the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f_1(x) + f_2(y),$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = g_1(x)\, g_2(y) + h(x).$$

What restrictions (if any) are required on the functions f_i, g_i and $h(x)$ for this system to pass Dulac's test with $\rho = 1$?

(b) Consider the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = g(y),$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(x)\, y - h(x).$$

What restrictions (if any) are required on the functions $f(x)$, $g(y)$ and $h(x)$ for this system to pass Dulac's test with $\rho = 1$?

(c) Find sufficient conditions for the system

$$\ddot{x} + \alpha\dot{x} + \beta x + x^2\dot{x} + x^3 = 0$$

to have no closed orbits. Are they also necessary? [39, exercise 1.8.3 on page 44]

4.7.3 Examples from research papers

1. In dimensionless form, the governing equations for the Belousov-Zhabotinsky reaction, as suggested by Tyson [78], are

$$\begin{aligned}
\epsilon_1 \frac{\mathrm{d}x}{\mathrm{d}\tau} &= \frac{(q-x)\, fz}{q+x} + x(1-x), \\
\frac{\mathrm{d}z}{\mathrm{d}\tau} &= x - z.
\end{aligned} \tag{4.2}$$

Here $x(\tau)$ and $z(\tau)$ represent the concentrations of $(HBr)\,O_2$ and Ce(UV) respectively. The parameter ϵ_1 is a dimensionless combination of some of the rate constants, the quantity q is related to the concentration of a certain chemical reagent and f is a stoichiometric parameter in one of the reactions.

The following questions are based upon [69]

(a) Write about the Belousov-Zhabotinsky reaction in one or two paragraphs. Provide references to books or web-pages you used as appropriate.

(b) Show that when $0 < q < 1$ that the region bounded by $q < x < 1$ and $0 < z < 1$ is positively invariant.

(c) Show that there is a critical value of ϵ, ϵ_{cr}, such that system (4.2) has no periodic solutions laying entirely within the invariant region for $\epsilon > \epsilon_{\mathrm{cr}}$.

(d) By considering the Dulac function

$$\rho(x, z) = B_0 x^{-(1-\epsilon)} (1-x)^{-(k+1+\epsilon_1)} \exp\left[\frac{kz}{\epsilon_1}\right],$$

where k is an adjustable parameter, show that there is a critical value of the parameter f, f_{cr}, such that for $f > f\mathrm{cr}$ system (4.2) has no periodic solutions laying entirely within the invariant region.

4.7.4 Extensions of the theory of Dulac's test

1. Explain in general terms how Dulac's test may be extended to cover the cases when

$$\frac{\partial \rho f}{\partial x} + \frac{\partial \rho g}{\partial y}$$

is of one sign except on isolated points or curves within a simply-connected region. [44, exercise 24 on page 91]

2. For the second-order system $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$, $\operatorname{div}(\mathbf{X}) = 0$ and $\operatorname{curl}(\mathbf{X}) = \mathbf{0}$ in a simply connected region $\mathcal{D}$. Show that the system has no closed paths in $\mathcal{D}$. Deduce that

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= y + 2xy,\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= x + x^2 - y^2\end{aligned}$$

has no limit cycles. [44, exercise 25 on page 91]

3. In exercise 2 show that $\operatorname{div}(\mathbf{X}) = 0$ may be replaced by $\operatorname{div}(\psi\mathbf{X}) = 0$ where $\psi(x, y)$ is of one sign in $\mathcal{D}$. [44, exercise 26 on page 92]

4. Let $\mathcal{D}$ be a doubly-connected domain in the (x, y) plane. Show that if $\rho(x, y)$ has continuous first partial derivatives and $\operatorname{div}(\rho\mathbf{X})$ is of constant sign in $\mathcal{D}$, then the system has not more than one closed path in $\mathcal{D}$. [44, exercise 30 on page 92]

5. Consider theorem 4.2. Show that if $\mathcal{A}$ is an annular region contained in $\mathcal{D}$ in which $\frac{\partial \rho f}{\partial x} + \frac{\partial \rho g}{\partial y}$ does not change sign, then the system has at most one limit cycle in $\mathcal{A}$.

   Hint. Assume that there are two limit cycles Γ_1 and Γ_2 in $\mathcal{A}$, connect them with a smooth arc Γ_0 (traversed in both directions), and then apply Green's Theorem to the resulting simply connected region whose boundary is $\Gamma_1 + \Gamma_0 - \Gamma_2 - \Gamma_0$. [62, problem 9.2 on page 263]

4.8 Things to do

1. Is it possible to have a periodic solution in a first-order ODE?

2. Look at assignment questions from week 9 (2008).

3. Nice to have a figure for the proof of Bendixson's criterion.

4. Need to put Green's theorem into an appendix.

5. Follow up [55, example 3.6 on page 155]

6. Jane's 1996b paper was cited in *ANZIAM Journal* **47**(4), 527–540, 2006.

   And in sexton:1996.

7. Dulac's criterion is discussed on page 305 of:

   Andronov, A.A., Vitt, E.A., and Khaiken, S.E. (1966). *Theory of Oscillations.* Pergamon Press: Oxford. (1949 edition of the book is 531.32/2)

   and in

   Andronov, A.A., Leontovich, E.A., Gordon, I.I., and A.G. Maier. (1973). *Qualitative Theory of Second-Order Dynamical Systems.* Wiley: New York. (not in library).