Chapter 6

Second-order differential equations: Periodic behaviour

6.1 Introduction

In this chapter we continue to restrict our attention to the case in which the system of differential equations consists of two first-order differential equations. In chapter 1 we showed, for the case of a scalar differential equation, that stability is lost when the eigenvalue of of a steady-state solution increases through zero. In chapter 3 we learnt, for a system of two differential equations, that stability can be lost by a similar mechanism: one eigenvalue changing from negative to positive, with a requirement that the second eigenvalue remains negative. In two (and more) dimensions there is an additional way for a steady-state solution to change stability: a pair of complex eigenvalues crossing the imaginary axis. This mechanism is illustrated in figure 6.1. When two complex eigenvalues cross the imaginary axis a *Hopf*[1] bifurcation occurs. The significance of this bifurcation is that it is associated with the formation, or destruction, of a limit cycle. The identification of a Hopf bifurcation in a system is the most common technique used to identify the existence of limit cycles.

Question 6.1 (Linear systems review) *Consider the linear system*

$$\dot{x} = -\omega y,$$
$$\dot{y} = \omega x.$$

1. *Determine the stability of the steady-state solution $x = y = 0$ by linearised stability analysis.*
2. *Show that this system has an infinite number of period solutions but no limit cycles. What does this tell you about the stability of the origin?*

Linear systems with constant coefficients cannot exhibit limit cycles. Since normally we cannot solve systems of nonlinear differential equations, which would provide an easy way to check if the system has a limit cycle, it is important to have other means of determining whether a limit cycle is present. In this chapter we describe a mechanism by which limit cycles 'bifurcate' from a steady-state solution when the steady-state solution changes its stability as two complex eigenvalues cross the imaginary axis. The bifurcation of a limit cycle from a steady-state solution in this manner is known as a Hopf bifurcation.

We start in section 6.2 by considering two motivating examples that can be completely analysed by changing from cartesian to polar coordinates. These examples represent what are called 'supercritical' and 'subcritical' Hopf bifurcations. In section 6.3 we state, and discuss, the Hopf bifurcation theorem. Essentially, a Hopf

[1] Eberhard Hopf (1902–1983) was an Austrian mathematician who made significant contributions in topology and ergodic theory. He also did important work in hydrodynamics, the theory of turbulence and radiative transfer theory. Source `www-groups.dcs.st-and.ac.uk/~history/Biographies/Hopf_Eberhard.html`.

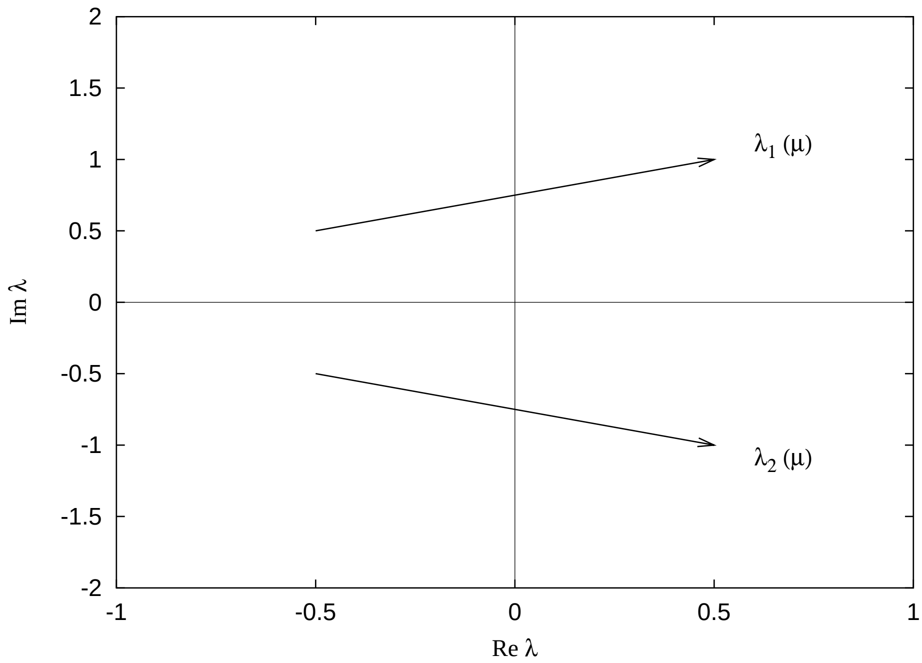


Figure 6.1: Schematic diagram showing the eigenvalues crossing the imaginary axis as a function of the bifurcation parameter μ. (The eigenvalues could equally well cross the imaginary axis in the opposite direction.)

bifurcation occurs when two complex eigenvalues cross the imaginary axis. The Hopf bifurcation theorem introduces a slightly mysterious quantity: l_1; the first Liapunov coefficient. The first Liapunov coefficient is defined in section 6.4. In section 6.5 we show how the sign of the first Liapunov coefficient is related to whether a Hopf bifurcation is subcritical or supercritical. In section 6.6 we give the 'normal form' for the Hopf bifurcation and briefly discuss the significance of the 'normal form'. Finally, in section 6.7 we indicate how the presence of Hopf bifurcation points and limit cycles are indicated on a steady-state diagram.

6.2 Motivation

Consider the following planar system that depends on one parameter

$$\begin{aligned} \dot{x} &= \mu x - y - x\left(x^2 + y^2\right), \\ \dot{y} &= x + \mu y - y\left(x^2 + y^2\right). \end{aligned} \tag{6.1}$$

In polar coordinates (ρ, θ) this system takes the form

$$\begin{aligned} \dot{\rho} &= \left(\mu - \rho^2\right)\rho, \\ \dot{\theta} &= 1. \end{aligned} \tag{6.2}$$

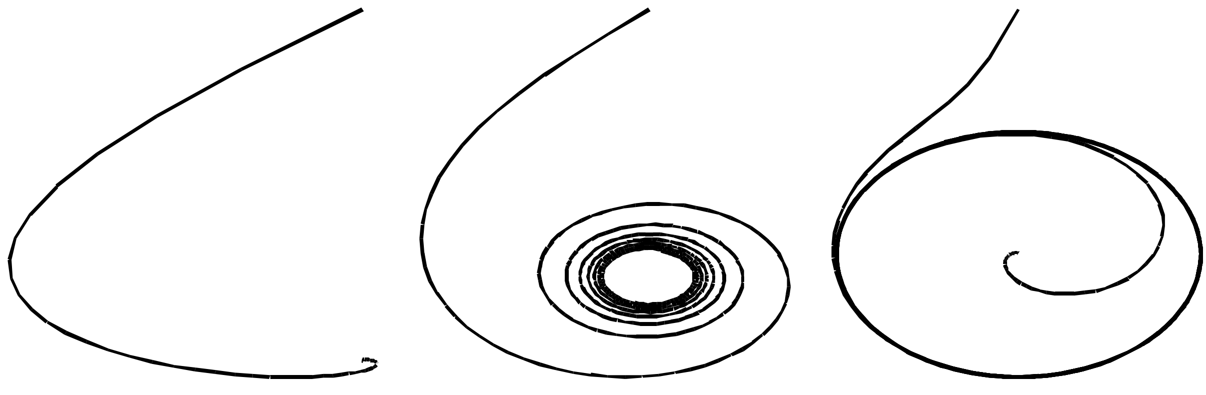
This system has the trivial solution $\rho = 0$ for all values of μ, and a solution $\rho = \sqrt{\mu}$ for $\mu > 0$. The solution of the second differential equation, $\dot{\theta} = 1$, $\theta = t + c$, represents a rotation with constant speed. (Note that the first differential equation can also be integrated explicitly). The steady-state solution $\rho = \sqrt{\mu}$, $\mu > 0$, therefore represents a circle in the $\rho - \theta$ plane and a limit cycle in the $x - y$ plane.

The following statements follow quite easily for system 6.2.

- For $\mu \leq 0$, the trivial steady-state solution is globally asymptotically stable, since $\dot{\rho} < 0$ for all values of ρ except $\rho = 0$.
- For $\mu > 0$, one has $\dot{\rho} > 0$ for small $\rho > 0$ so that the trivial steady-state solution is linearly unstable for any $\mu > 0$.

- For $\mu > 0$, system (6.2) has a *limit cycle* for any $\mu > 0$ of radius $\rho_0 = \sqrt{\mu}$ (at $\rho = \rho_0$ we have $\dot{\rho} = 0$). Moreover, the limit cycle is globally asymptotically stable, since $\dot{\rho} > 0$ inside and $\dot{\rho} < 0$ outside the cycle.

As the bifurcation parameter μ increases through zero, the trivial steady-state solution loses stability and a stable limit cycle 'bifurcates' from the steady-state solution (the origin). This is referred to as a Hopf bifurcation, or sometimes the Andronov[2]-Hopf bifurcation, as in [18], or the Poincaré[3]-Andronov-Hopf bifurcation; see the comments in section 6.8.2. In system 6.1 this leads to the appearance, from the steady-state solution, of small-amplitude periodic solutions. Since the equations for ρ and θ are independent in system (6.2), one can easily draw phase portraits of the system (see figure 6.2).



(a) $\mu < 0$ (b) $\mu = 0$ (c) $\mu > 0$

Figure 6.2: Hopf bifurcation. (After Kuznetsov, 1995). Arrows showing the direction of increasing time not shown.

Question 6.2 (Supercritical Hopf bifurcation)

1. *Consider the system* (6.1).
   (a) *Show that the only equilibrium of this system is* $x_1 = x_2 = 0$ *for all* μ *with the Jacobian matrix*
   $$A = \begin{pmatrix} \mu & -1 \\ 1 & \mu \end{pmatrix}.$$
   (b) *Show that the equilibrium* $x_1 = x_2 = 0$ *has purely imaginary eigenvalues at* $\mu = 0$.
   (c) *Show that for* $\mu > 0$ *a solution of equations* (6.1) *is given by*
   $$x(t) = \sqrt{\mu}\cos t,$$
   $$y(t) = \sqrt{\mu}\sin t.$$
   (d) *Convert the Cartesian form of the equations* (6.1) *to the polar form* (6.2).
2. *Consider the system* (6.2). *Show that the equilibrium point* $\rho = 0$ *is*

[2]A.A. Andronov (1901–1952), a Soviet physicist and an Academician of the USSR Academy of Sciences (1946). A key figure in the development of control engineering in the former Soviet Union. One of his great achievements was to realise how oscillatory phenomena in chemistry, biology and engineering could be explained by phase-plane techniques. Founder of a scientific school on the theory of non-linear oscillations and dynamical systems. Sources: `ict.open.ac.uk/reports/1.pdf` and [5, page 192].

[3]Henri Poincaré (1854–1912). French mathematician, regarded as a 'giant' of 19th century mathematics and one of the greatest mathematicians of all time. He was one of the last 'universal mathematicians', i.e. one who had a general grasp of all branches of mathematics. He transformed many branches of mathematics and contributed greatly to mathematical physics, especially celestial mechanics. His work in celestial mechanics, sometimes considered to be the biggest advance in the subject since Newton's work, lead to the first mathematical description of chaotic behaviour in a dynamical system. For more details on his life see [29, pages 237–245].

   (a) *Stable if* $\mu < 0$.
   (b) *Stable if* $\mu = 0$.
   (c) *Unstable if* $\mu > 0$.

3. *Show that the equilibrium* $\rho_0(\mu) = \sqrt{\mu}$ *is linearly stable for* $\mu > 0$. *Explain why this solution represents an isolated closed orbit (or limit cycle) of system* (6.2).
4. *Show that for* $\mu > 0$ *any initial condition (except* $\rho(0) = 0$*) outside or inside the cycle* $\rho_0(\mu) = \sqrt{\mu}$ *tend to the cycle as* $t \to \infty$.

The type of Hopf bifurcation that occurs in this example is known as a *supercritical* Hopf bifurcation because for values of the primary bifurcation parameter, μ, slightly *larger* than the critical value, $\mu_{\text{cr}} = 0$, there are 'small' amplitude limit cycles 'near' the steady-state solution from which the limit cycles have bifurcated. We can view the limit cycles as being 'created' at the Hopf bifurcation point as the primary bifurcation parameter is increased through the value $\mu = 0$ or as being 'destroyed' at the Hopf bifurcation point as the primary bifurcation parameter is decreased through the value $\mu = 0$.

Question 6.3 (Subcritical Hopf Bifurcation) *Consider a system having nonlinear terms with the opposite sign to system* (6.1).

$$\begin{aligned} \dot{x} &= \mu x - y + x\left(x^2 + y^2\right) \\ \dot{y} &= x + \mu y + y\left(x^2 + y^2\right). \end{aligned} \tag{6.3}$$

Analyse this in the same way as system (6.1) *and show that*

1. *The system undergoes a Hopf bifurcation at* $\mu = 0$.
2. *The system has an unstable limit cycle which disappears when* μ *crosses zero from negative to positive values.*
3. *The equilibrium at the origin is*
   (a) *Stable for* $\mu < 0$.
   (b) *Unstable at* $\mu = 0$.
   (c) *Unstable for* $\mu > 0$.

The type of Hopf bifurcation that occurs in this example is known as a *subcritical* Hopf bifurcation because for values of the primary bifurcation parameter, μ, slightly *smaller* than the critical value, $\mu_{\text{cr}} = 0$, there are 'small' amplitude limit cycles 'near' the steady-state solution from which the limit cycles have bifurcated. We can view the limit cycles as being 'destroyed' at the Hopf bifurcation point as the primary bifurcation parameter is increased through the value $\mu = 0$ or as being 'created' at the Hopf bifurcation point as the primary bifurcation parameter is decreased through the value $\mu = 0$.

In both of the examples considered in this section the steady-state solution losses stability as the primary bifurcation parameter is increased through the value $\mu = 0$. In the first case, the supercritical Hopf bifurcation, the stable equilibrium is replaced by a stable limit cycle of small amplitude. Therefore, the system "remains" in a neighbourhood of the equilibrium and we have a *soft* stability loss. In the second case, the subcritical Hopf bifurcation, the basin of attraction of the steady-state solution is bounded by the unstable cycle, which "shrinks" as the parameter approaches its critical value and disappears. Thus, the system is "pushed out" from a neighbourhood of the equilibrium, giving us a *sharp* loss of stability.

Is this distinction between 'soft' and 'hard' loss of stability important? If a system loses stability softly the long-time solution changes by a small amount as the limit cycles have small amplitude. Furthermore, the system is "controllable": if we make the primary bifurcation parameter negative, the system returns to the stable equilibrium. On the contrary, if the system loses stability sharply, the long-time solution no longer changes by a small amount - there is no solution 'near' the unstable steady-state solution. (In fact system 6.3 does not have a stable solution for $\mu > 0$). Furthermore, the system is not well "controllable": if we make the primary bifurcation parameter negative, the system may not return back to the stable equilibrium since it may have left its region of attraction.

6.3 The Hopf bifurcation theorem

Generically, a Hopf bifurcation occurs when a pair of complex eigenvalues crosses the imaginary axis. Associated with this it the formation (or destruction) of a limit cycle.

Theorem 6.1 (Hopf bifurcation theorem) *Suppose a two-dimensional system*

$$\frac{dx}{dt} = f(x, \mu), \; x \in \mathbb{R}^2, \; \mu \in \mathbb{R}, \tag{6.4}$$

with $f \in C^2$, has a steady-state solution $\mathbf{x}(\mu)$ with imaginary eigenvalues

$$\lambda_{1,2}(\mu) = \mu(\mu) \pm i\omega(\mu)$$

which are purely imaginary at $\mu = \mu_0$, i.e. $\mu(\mu_0) = 0$ and $\omega(\mu_0) = \omega_0 > 0$.

Let the following non-degeneracy conditions be satisfied:

(1) The derivative of the real part of the eigenvalues with respect to the bifurcation parameter, μ, is non-zero, i.e.

$$\left.\frac{d}{d\mu} Re\,\lambda_{1,2}(\mu)\right|_{\mu=0} \neq 0.$$

(2) $l_1(0) \neq 0$, where l_1 is known as the first Lyapunov coefficient.

Then it can be shown that a unique limit cycle bifurcates from the steady-state solution at the Hopf bifurcation point $(\mathbf{x}, \mu) = (\mathbf{0}, \mu_0)$. The initial period of the zero-amplitude oscillation is

$$T_0 = \frac{2\pi}{\omega(0)}.$$

We return to the mysterious Liapunov coefficient in section 6.4

Remark 6.1 *The first condition means that the pair of complex-conjugate eigenvalues $\lambda_{1,2}(\mu)$ crosses the imaginary axis with nonzero speed. This is sometimes called the 'transversality condition' and is illustrated in figure 6.1. The second condition implies that a certain combination of Taylor coefficients of the right-hand side of the system (up to and including third-order coefficients) does not vanish. An explicit formula for $l_1(0)$ can be obtained — see section 6.4.*

Remark 6.2 *At the critical value of μ, μ_0, the linearised system has a pair of conjugate imaginary eigenvalues and has an infinite number of periodic solutions. (See question 6.1).*

Remark 6.3 *For a planar system, a limit cycle must 'surround' a steady-state solution.*

It can also be shown that if z is some measure of the amplitude of the periodic solution then along the periodic solution branch the value of the primary bifurcation parameter, μ, can be expressed as an even function of the amplitude of the periodic solution, i.e.

$$\mu - \mu_{\text{cr}} = l_1 z^2 + l_2 z^4 + \dots,$$

where μ_{cr} is the value of the bifurcation parameter at the Hopf bifurcation, l_1 is the first Liapunov coefficient, l_2 is the second Liapunov coefficient etc. Note, near the Hopf bifurcation point where the amplitude is small this implies that the amplitude is given by

$$z \approx \sqrt{\frac{\mu - \mu_{\text{cr}}}{l_1}}.$$

Question 6.4 *Show that for a second-order system that a steady-state solution has purely imaginary eigenvalues at the critical value for μ ($\mu = \mu_0$) when: (i) the trace of the Jacobian matrix is equal to zero; (ii) the determinant of the Jacobian matrix is positive;*

6.4 The first Liapunov coefficient

In section 6.3 we stated the Hopf bifurcation system. One of the non-degeneracy conditions is that the first Liapunov coefficient must be non-zero. However, in section 6.3 we did not define the first Liapunov coefficient. We do so in this section.

Theorem 6.2 (First Liapunov coefficient, from [47], page 344) *For a planar analytic system*

$$\dot{x} = ax + by + p(x,y),$$
$$\dot{y} = cx + dy + q(x,y),$$

with $\Delta = ad - bc > 0$, $a + d = 0$ *and* $p(x,y)$, $q(x,y)$ *given by the series*

$$\begin{aligned} p(x,y) &= \sum_{i+j\geq 2} a_{ij}x^i y^j, \\ &= \left(a_{20}x^2 + a_{11}xy + a_{02}y^2\right) + \left(a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3\right) + \ldots, \\ q(x,y) &= \sum_{i+j\geq 2} b_{ij}x^i y^j, \\ &= \left(b_{20}x^2 + b_{11}xy + b_{02}y^2\right) + \left(b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3\right) + \ldots \end{aligned}$$

the matrix

$$D\mathbf{f}(\mathbf{0}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

will have a pair of imaginary eigenvalues. The first Liapunov coefficient $l_1(0)$ *is then given by the formula*

$$\begin{aligned} l_1(0) = \frac{-3\pi}{2b\Delta^{3/2}} &\left\{\left[ac\left(a_{11}^2 + a_{11}b_{02} + a_{02}b_{11}\right) + ab\left(b_{11}^2 + a_{20}b_{11} + a_{11}b_{02}\right) + c^2\left(a_{11}a_{02} + 2a_{02}b_{02}\right)\right.\right. \\ &-2ac\left(b_{02}^2 - a_{20}a_{02}\right) - 2ab\left(a_{20}^2 - b_{20}b_{02}\right) - b^2\left(2a_{20}b_{20} + b_{11}b_{20}\right) + \left(bc - 2a^2\right)\left(b_{11}b_{02} - a_{11}a_{20}\right)\big], \\ &\left.- \left(a^2 + bc\right)\left[3\left(cb_{03} - ba_{30}\right) + 2a\left(a_{21} + b_{12}\right) + \left(ca_{12} - bb_{21}\right)\right]\right\}. \end{aligned} \tag{6.5}$$

In principle it is a straightforward, if tedious, calculation to use equation (6.5) to determine the first Liapunov coefficient. In practice the calculation cab be "*algebraically hairy*" [15, page 281]. This is a calculation that is well-suited for the capabilities of Maple, see section 6.9.

What is the significance of the first Liapunov coefficient? Partly it shows that one of the non-degeneracy conditions in the Hopf bifurcation theorem is satisfied. More importantly the sign of the first Liapunov coefficient partly determines whether a Hopf bifurcation is subcritical or supercritical.

Under special conditions equation (6.5) can be considerably simplified.

Question 6.5 (from [47], page 343) *Consider the planar analytic system*

$$\begin{aligned} \dot{x} &= \mu x - y + p(x,y), \\ \dot{y} &= x + \mu y + q(x,y), \end{aligned} \tag{6.6}$$

where the functions $p(x,y)$ *and* $q(x,y)$ *are given by the series in theorem 6.2.*

1. *Show that the steady-state* $x = y = 0$ *undergoes a Hopf bifurcation at the parameter value* $\mu = 0$.
2. *Show that the Liapunov number simplifies to*

$$\sigma = \frac{3\pi}{2}\left[3\left(a_{30} + b_{03}\right) + \left(a_{12} + b_{21}\right) - 2\left(a_{20}b_{20} - a_{02}b_{02}\right) + a_{11}\left(a_{02} + a_{20}\right) - b_{11}\left(b_{02} + b_{20}\right)\right]. \tag{6.7}$$

A fair question to ask here is 'why question 6.5'? What is the usefulness of equation 6.7 when we have equation 6.5? The linear part of system 6.6 is in the 'normal form' for the hopf bifurcation and we see that when this happens the calculation of the first Liapunov coefficient dramatically simplifies. (The 'normal form' for the Hopf bifurcation is given in section 6.6).

6.5 Sub-critical and super-critical Hopf bifurcations

We know from the analysis in section 6.2 that a Hopf bifurcation occurs in the systems (6.1) & (6.3) at the origin when the bifurcation parameter takes value $\mu = 0$. The former was called a 'supercritical' Hopf bifurcation whilst the latter was called a 'subcritical' Hopf bifurcation. By 'supercritical' we mean that small amplitude limit cycles are found as the bifurcation parameter is *increased* through the critical value whereas by 'subcritical' we mean that small amplitude limit cycles are found as the bifurcation parameter is *decreased* through the critical value. In this section we see how knowledge of the sign of the first Liapunov coefficient helps to determine whether a Hopf bifurcation is subcritical or supercritical.

An alternative definition of subcritical and supercritical Hopf bifurcation is that in the supercritical case a stable limit cycle co-exists with an unstable steady-state whilst in the subcritical case an unstable limit cycle co-exists with a steady-state stable equilibrium. In this case the definition is independent of the direction in which the primary bifurcation parameter is being changed. These definitions of super- and sub-critical Hopf bifurcations agrees with the examples used in section 6.2.

Remark 6.4 *In a planar system the steady-state solution must change stability on either side of a Hopf bifurcation. If the system contains three, or more, differential equations this is no longer true. The alternative definition given above only applies if the steady-state solution changes stability on either side of the Hopf bifurcation.*

Question 6.6 *Explain why*

1. *In a planar system the stability of a steady-state solution must differ on either side of a Hopf bifurcation.*
2. *If there are three, or more, equations in the system the steady-state stability does not have to change on either side of a Hopf bifurcation.*

Theorem 6.3 (Hopf bifurcation and Liapunov number from [47], page 343–344.) *Assume that the conditions of theorem 6.1 hold.*

(1) Suppose that

$$\frac{d}{d\mu} Re\, \lambda_{1,2}(\mu)\bigg|_{\mu=0} > 0.$$

Then if $l_1 < 0$ a unique stable limit cycle bifurcates from the steady-state solution as μ increases through the Hopf bifurcation point. If $l_1 > 0$, then a unique unstable limit cycle bifurcates from the steady-state solution as μ decreases through the Hopf bifurcation point.

(2) Suppose that

$$\frac{d}{d\mu} Re\, \lambda_{1,2}(\mu)\bigg|_{\mu=0} < 0.$$

Then if $l_1 < 0$ a unique stable limit cycle bifurcates from the steady-state solution as μ decreases through the Hopf bifurcation point. If $l_1 > 0$, then a unique unstable limit cycle bifurcates from the steady-state solution as μ increases through the Hopf bifurcation point.

Example 6.1 (Supercritical Hopf bifurcation) *Consider the system [25, page 270]*

$$x' = \mu x + y,$$
$$y' = -x + \mu y - x^2 y$$

The origin is a steady-state solution of this system. The eigenvalues of the trivial solution are given by

$$\lambda_{1,2} = \mu \pm \sqrt{-1}.$$

It now follows that

1. *When $\mu = 0$ the eigenvalues are purely imaginary.*
2. *When $\mu = 0$ we have*
$$\frac{d}{d\mu} Re\,\lambda_{1,2}(\mu)\bigg|_{\mu=0} = 1 > 0.$$
Thus the first part of theorem 6.3 applies.

We now need to calculate the Liapunov number. We have $a = 0$, $b = 1$, $c = -1$, $d = 0$ and thus that $\Delta = 1$. Furthermore the functions p, $p = 0$, and q, $g = -x^2 y$, are particularly nice polynomials and all of the coefficients required to calculate l_1 are zero except for b_{21} which is given by $b_{21} = -1$. It now follows that

$$l_1 = -\frac{3}{2}\pi < 0$$

It now follows from theorem 6.3 that as the bifurcation parameter μ is increased through zero a unique stable limit cycle bifurcates from the steady-state solution, i.e. there is soft generation of limit cycles.

In example 6.1 it was easy to calculate the sign of l_1. This is not usually the case, and it is useful to write some maple code to automate the calculation of the first Liapunov number. Sample code to do this is provided in section 6.9.

Example 6.2 (Change of coordinates) *Consider the system [25, page 271]*

$$\begin{aligned} x' &= x(\mu - x) - (x+1)\,y^2, \\ y' &= y(x-1). \end{aligned} \tag{6.8}$$

It can be shown that the steady-state solutions of this system are given by

$$\begin{aligned} (x_1, y_1) &= (0, 0), \\ (x_2, y_2) &= (\mu, 0), \\ (x_{3,4}, y_{3,4}) &= (1, y_\pm) \end{aligned}$$

where $y_\pm$ is given by

$$y_\pm^2 = \frac{1}{2}(\mu - 1).$$

It can further be shown that the solution branch $(x_{3,4}, y_\pm)$ is stable for $1 < \mu < 3$, unstable for $3 > \mu$ and that it loses stability through a Hopf bifurcation at $\mu = 3$. We would like to know whether the emerging limit cycle is supercritical and stable or subcritical and unstable. However, we can not apply theorem 6.3 because the steady-state solution is not located at the origin.

We therefore define new variables

$$\begin{aligned} X &= x - 1, \\ Y &= y - \epsilon, \\ \mu &= 1 + 2\epsilon^2. \end{aligned}$$

In these news variables system 6.8 becomes

$$\begin{aligned} X' &= \left(\epsilon^2 - 1\right) X - 4\epsilon Y - X^2 - 2\epsilon XY - 2Y^2 - XY^2, \\ Y' &= \epsilon X + XY. \end{aligned}$$

The origin is now a steady-state solution of this system. The Jacobian matrix is given by

$$J(\epsilon) = \begin{pmatrix} \epsilon^2 - 1 & -4\epsilon \\ \epsilon & 0 \end{pmatrix}$$

with eigenvalues

$$\lambda_{1,2} = \frac{\left(\epsilon^2 - 1\right) \pm \sqrt{\epsilon^4 + 1 - 6\epsilon^2}}{2}.$$

We deduce that:

1. *when* $\epsilon = 1$ *the eigenvalues are purely imaginary;*

2. *when* $\epsilon = 1$ *we have*
$$\frac{d}{d\epsilon} Re\,\lambda_{1,2}(\epsilon)\bigg|_{\epsilon=1} = 1 > 0.$$
Thus the first part of theorem 6.3 applies.

We now calculate the Liapunov number. When $\epsilon = 1$ *we have* $a = 0$, $b = -4$, $c = 1$ *and* $d = 0$. *The functions* p *and* q *are given by*

$$p = -X^2 - 2XY - 2Y^2 - XY^2,$$
$$q = XY.$$

Most of the coefficients required to calculate l_1 *are zero and it is straightforward to calculate the value for* l_1 *by hand. However, it is even easier if the code in section 6.9 is available! We obtain*

$$l_1 = 0.375\pi > 0.$$

It now follows from theorem 6.3 that a unique unstable limit cycle bifurcates from the steady-state solution as ϵ *(*μ*) is decreased through one (three), i.e. there is a strong generation of stability as the bifurcation parameter is increased through* $\mu = 3$.

6.6 'Normal form' for the Hopf bifurcation

What is a 'normal form'? Broadly speaking a 'normal form' is a simplification of an object down to its essential 'bare bones'. A 'normal form' retains the essential features of an object whilst discarding anything that is non-essential. Normal forms are often obtained by applying a clever change of co-ordinates and are important in the study of qualitative dynamics, unfoldings and bifurcations. [4]

Theorem 6.4 says that any system satisfying theorem 6.1 can be 'manipulated' (in a well defined way) to either system 6.1 or system 6.3. For instance, the location of the steady-state solution, $\mathbf{x}(\mu)$, can always be moved to the origin , $\mathbf{x} = \mathbf{0}$, by a coordinate shift. Similarly, the critical value of the primary bifurcation parameter, $\mu = \mu_{\text{cr}}$, can always be shifted to $\mu_0 = 0$. The 'normal form' approximates the original system only for values of (x, y) near the origin and for values of the primary bifurcation parameter near the critical value.

Theorem 6.4 (Topological normal form for the Hopf bifurcation) *If the conditions specified in theorem 6.1 hold then the system* (6.4) *is locally topologically equivalent near the Hopf bifurcation point to one of the following normal forms:*

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}\mu & -1\\ 1 & \mu\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} + \sigma\left(x^2+y^2\right)\begin{pmatrix}x\\ y\end{pmatrix}, \tag{6.9}$$
$$\sigma = sign\, l_1(0). \tag{6.10}$$

Observe that the normal form of the Hopf bifurcation 6.9 contains no quadratic terms (x^2, y^2, xy) and no mixed cubic terms (x^2y, xy^2). Furthermore, the coefficients in front of the remaining cubic terms are identical.

The normal form for the Hopf bifurcation is derived in [31, chapter 3.5].

[4] This description of a normal form is based on the Scholarpedia entry on normal forms. `www.scholarpedia.org/article/Normal_form`.

6.7 Steady-state diagrams

In steady-state diagrams it is common to mark a Hopf bifurcation point by a filled in square (■). Stable and unstable periodic solutions are marked by filled (•) and unfilled circles (∘) respectively. Suppose that the periodic solution is available in the form $(x(t), y(t))$, usually obtained numerically, and that the steady-state diagram will plot the x component as a function of the primary bifurcation parameter (μ). In plotting information about the periodic solution there are various options.

1. One can plot the maximum value of the solution component, i.e. $x_{\max}$, as a function of the primary bifurcation parameter μ.
2. One can plot the maximum and minimum value of a component of the solution, e.g. plot $x_{\max}$ and $x_{\min}$, as a function of the primary bifurcation parameter μ.
3. Suppose that the periodic-solution has period T. One can plot the integral norm $(\mathcal{I})$ of a component of the solution, e.g.
$$\mathcal{I} = \frac{1}{T}\int_0^T x(t)\,\mathrm{d}t$$

This is not an exhaustive list.

Question 6.7 *Draw a steady-state diagram showing x^* as a function of μ for the system considered in questions* (6.2) *&*(6.3)*. Use either option 1 or option 2 from above.*

6.8 Conclusions

6.8.1 Summary

In this chapter we have learnt that the Hopf bifurcation acts as a mechanism for the generation of limit cycles. A Hopf bifurcation occurs when a complex pair of eigenvalues passes through the imaginary axis and can be categorised as being either supercritical or subcritical, depending upon if small amplitude limit cycles exists either 'before' or 'after' the Hopf bifurcation point. (This classification assumes that the primary bifurcation parameter is being increased through the critical value). The specific criticality of a Hopf bifurcation point is determined by the sign of the 'velocity' of the real part of the eigenvalues as they cross the imaginary axis and by the sign of the first Liapunov coefficient at the Hopf bifurcation point — see theorem 6.3 for the details.

6.8.2 Historical remarks

Theorem 6.1 was proved by Hopf for the n-dimensional case in 1942 [28], with the additional assumption that are no other eigenvalues with zero real part at the critical value of the bifurcation parameter. (This assumption is not needed for a planar set of differential equations — why?). An English-language translation of Hopf's paper is included in [36].

However, the 'Hopf' bifurcation first appeared 50 years earlier in Poincaré's work and was analysed for second-order systems by Andronov and Leontovich[5] in 1939 [3]. It is therefore sometimes called the Poincaré–Andronov–Hopf bifurcation.

An explicit expression for the first Lyapunov coefficient in terms of Taylor coefficients of a general planar system was obtained by Bautin [11][6]. An exposition of the results by Andronov and his co-workers on the 'Hopf'-bifurcation can be found in [4].

(Historical remarks heavily based upon [31, page 102]).

[5]Evgeniya Leontovich (1905–1996) was Andronov's wife.
[6]Nikolay Bautin. A collaborator and former student of Andronov's.

6.9 Maple commands

The following code is very useful for determining the first Liapunov number. It can only be used for systems which are already in the form required for theorem 6.3.

```
# liapunov    Calculates the first liapunov number for a planar
# 24.04.08    system of differential equations.
#
# define the first liapunov number
l1 := (-3*Pi)/(2*b*detJ^1.5)*( (a*c*(a11^2+a11*b02 +a02*b11) \
     +ab*(b11^2+a20*b11+a11*b02) +c^2*(a11*a02+2*a02*b02) \
     -2*a*c*(b02^2-a20*a02) -2*a*b*(a20^2-b20*b02) -b^2*(2*a20*b20+b11*b20) \
     +(b*c-2*a^2)*(b11*b02-a11*a20)) -(a^2+b*c)*( 3*(c*b03-b*a30) \
     +2*a*(a21+b12) +(c*a12-b*b21)));

# define the linear coefficients at the Hopf bifurcation point
# and calculate the determinant.
a    := 0:
b    := 1:
c    := -1:
d    := 0:
detJ := a*d -b*c;

# define the functions p and q
p := 0;
q := -x^2*y;

# calculate the required coefficients.
a20 := coeftayl(p,[x,y]=[0,0],[2,0]);
a11 := coeftayl(p,[x,y]=[0,0],[1,1]);
a02 := coeftayl(p,[x,y]=[0,0],[0,2]);
a30 := coeftayl(p,[x,y]=[0,0],[3,0]);
a21 := coeftayl(p,[x,y]=[0,0],[2,1]);
a12 := coeftayl(p,[x,y]=[0,0],[1,2]);
a03 := coeftayl(p,[x,y]=[0,0],[0,3]);
b20 := coeftayl(q,[x,y]=[0,0],[2,0]);
b11 := coeftayl(q,[x,y]=[0,0],[1,1]);
b02 := coeftayl(q,[x,y]=[0,0],[0,2]);
b30 := coeftayl(q,[x,y]=[0,0],[3,0]);
b21 := coeftayl(q,[x,y]=[0,0],[2,1]);
b12 := coeftayl(q,[x,y]=[0,0],[1,2]);
b03 := coeftayl(q,[x,y]=[0,0],[0,3]);

# We are now ready to calculate the liapunov number
l1;
```

6.10 Revision of key ideas

The following questions are about the key ideas in this chapter.

1. Can a linear planer system have either limit cycles or periodic solutions?
2. Describe the mechanism by which a Hopf bifurcation occurs.

3. What is meant by the phrases 'supercritical' Hopf bifurcation and 'subcritical' Hopf bifurcation?
4. What conditions must hold for a Hopf bifurcation to occur in a planar system¿
5. State the two non-degeneracy conditions that are associated with the Hopf bifurcation theorem and explain what they mean.
6. What is the initial period of the limit-cycles generated by a Hopf bifurcation?
7. At a Hopf bifurcation a limit cycle bifurcates from a steady-state solution. How does the amplitude of the limit cycle vary for values of the primary bifurcation parameter near the critical value?
8. What is the significance of the first Liapunov coefficient?
9. What determines whether a Hopf bifurcation is subcritical or supercritical?
10. How is a Hopf bifurcation point represented on a steady-state diagram?
11. How is information about a limit cycle represented on a steady-state diagram?

6.10.1 Questions on the Hopf bifurcation

1. Calculate the Liapunov number at the Hopf bifurcation in:
   (a) system (6.1),
   (b) system (6.3).
2. Check that each of the following systems has an equilibrium point that exhibits the Hopf bifurcation at some value of α, and compute the first Lyapunov coefficient.
   (a) *Rayleigh's equation*
   $$\ddot{x} + \dot{x}^3 - 2\alpha\dot{x} + x = 0$$
   (Hint: Introduce $y = \dot{x}$ and rewrite the equation as a system of two differential equations.)
   (b) *Van der Pol's oscillator*:
   $$\ddot{y} - \left(\alpha - y^2\right)\dot{y} + y = 0;$$
   (c) *Bautin's example*
   $$\begin{aligned}\dot{x} &= y\\ \dot{y} &= -x + \alpha y + x^2 + xy + y^2;\end{aligned}$$
   (d) *Brusselator*:
   $$\begin{aligned}\dot{x} &= a - (\alpha+1)\,x + x^2 y\\ \dot{y} &= \alpha x - x^2 y;\end{aligned}$$
   (e) *Advertising diffusion model* [19]
   $$\begin{aligned}\dot{x} &= \alpha\left[1 - xy^2 + A\,(y-1)\right]\\ \dot{y} &= xy^2 - y.\end{aligned}$$

   [Kuznetsov, Chapter 3.6, question 6]
3. Hodgkin and Huxley received the Nobel Prize in Physiology and Medicine, with Eccles, in 1963 for developing the modern understanding of nerve conduction. Their model, based upon experimental work consists of three ordinary differential equations and one partial differential equation. FitzHugh reduced the four-variable Hodgkin-Huxley model to the following planar system.
   $$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= c\left(y + x - \frac{1}{3}x^3 - I\right),\\ c\frac{\mathrm{d}y}{\mathrm{d}t} &= a - x - by.\end{aligned}$$

Here x is an excitability variable, such as membrane potential, y is a recovery variable such as potassium channel activity, I is an input, such as applied content, (which will be assumed constant), and a, b and c are positive parameters, and $b < c$, $b < 1$, $b < c^2$.

(a) Verify that the Jacobian matrix at a steady-state (x^*, y^*) is given by

$$J = \begin{pmatrix} c\left(1 - x^{*^2}\right) & c \\ -\frac{1}{c} & -\frac{b}{c} \end{pmatrix},$$

and that the steady-state is stable if

$$\frac{b}{c} - c\left(1 - x^{*^2}\right) > 0, \quad 1 - b\left(1 - x^{*^2}\right) > 0.$$

(b) Show that the steady-state of the model is unstable if x^* falls in the range $-\gamma < x^* < \gamma$, where $\gamma = \sqrt{1 - \frac{b}{c^2}a}$.

(c) Show that if $I = I_{\text{cr}}$, where $I_{\text{cr}} = (a - \gamma)/b + \gamma - \gamma^3/3$, there is a steady-state solution at $(\gamma, (a - \gamma)/b)$, and that the Jacobian matrix there has purely imaginary eigenvalues.

(d) Show that if I increases from I_{cr}, the eigenvalues at the steady-state move into the right half plane, destabilising the steady-state, i.e. a Hopf bifurcation occurs.

[15, exercise 6.7 on pages 196–197]

4. For each of the following planar systems show that there is a Hopf bifurcation from the zero solution $x = y = 0$ at $\mu = 0$ and determine the nature of the bifurcating solution using theorem 6.3.

(a) $x' = \mu x + y - xy^2$, $\quad y' = -x + \mu y$,

(b) $x' = \mu x + y - x^3$ $\quad y' = -x + \mu y = 2y^3$,

(c) $x' = \mu x + y - x^2$, $\quad y' = -x + \mu y + x^2$,

(d) $x' = \mu x + y - x^2 + xy^2$, $y' = -x + \mu y - y^2$,

(e) $x' = y + \mu x - \frac{1}{3}x^3$, $\quad y' = -x$, (another version of Van der Pol's equation)

(f) $x' = y + \mu x - y^2$, $\quad y' = -4x + \mu y + y^2$,

(g) $x' = y + \mu x - x^3$, $\quad y' = -x$,

(a–f) from [25, problems 4 & 5 on page 274]. (g) from [27, pages 227–228]

5. Find the Hopf bifurcation which occur in the variational van der Pol equations [26, exercise 3.4.4. on page 152]

$$\dot{u} = u - \sigma v - u\left(u^2 + v^2\right),$$
$$\dot{v} = \sigma u + v - v\left(u^2 + v^2\right) - \gamma.$$

6. (a) Show that the planar system

$$x' = r^2\left(\mu - r^2\right)x + \omega\left(r^2\right)y,$$
$$y' = r^2\left(\mu - r^2\right)y - \omega\left(r^2\right)x,$$

where $r^2 = x^2 + y^2$ and $\omega(0) = 1$ has a Hopf bifurcation at $x = y = 0$ when $\mu = 0$.
Hint. Transform to polar co-ordinates. [25, problem 6 on page 274]

(b) Show that for the planar system

$$\dot{x} = \mu x + y - xf(r),$$
$$\dot{y} = -x + \mu y - yf(r),$$

where $r = \sqrt{x^2 + y^2}$, $f(r)$ is continuous for $r \geq 0$, $f(0) = 0$ and $f(r) > 0$ for $r > 0$ that

(i) for $\mu < 0$ the origin is globally asymptotically stable. (Use a Liapunov function).
(ii) for $\mu = 0$ the origin is globally asymptotically stable. (Use a Liapunov function).
(iii) That the origin is unstable for $\mu > 0$.
(iv) A Hopf bifurcation occurs at $\mu = 0$.
(v) Is the Hopf bifurcation subcritical or supercritical?

[30, Theorem 12.1 on page 328]

(c) Consider the planar system

$$\ddot{x} + \left(x^2 + \dot{x}^2 - \mu\right)\dot{x} + x = 0.$$

(a) Show that the system has no periodic cycles for $\mu < 0$. (Use Dulac's test).
(b) Show that a Hopf bifurcation occurs at $\mu = 0$.
(c) Is the Hopf bifurcation subcritical or supercritical?
(d) Determine the stability of the origin when $\mu = 0$.

[30, Theorem 12.1 on page 328]

(d) Consider the planar system

$$\dot{x} = -\mu x + y + \frac{x}{1 + x^2 + y^2},$$
$$\dot{y} = x - \mu y + \frac{y}{1 + x^2 + y^2}.$$

Show that the equations display a Hopf bifurcation as $\mu > 0$ decreases through $\mu = 1$. Find the radius of the periodic solution for $0 < \mu < 1$. [30, Exercise 12 on page 358]

7. For the planar system

$$x' = 2(1 - \mu)x + x^2 - xy,$$
$$y' = x + \frac{1}{2}x^2 - y,$$

show that the critical point for which $y = x + \frac{1}{2}x^2$ and $x^2 = 4(1 - \mu)$ has a Hopf bifurcation at $x = 1$, $y = \frac{3}{2}$ and $\mu = \frac{3}{4}$. Use theorem 6.3 to determine the form of the bifurcating periodic solution near the bifurcation point.

Hint. First make (and justify) the transformation $X = x - \epsilon$, $Y = y - \epsilon - \frac{1}{2}\epsilon^2$ and $\epsilon^2 = 4(1 - \mu)$. [25, problem 7 on page 274]

8. In the Duffing equation

$$\ddot{x} + \mu\dot{x} + \left(x - x^3\right) = 0,$$

a bifurcation with pure imaginary eigenvalues occurs at $\mu = 0$ when the system changes from having negative to positive dissipation, but it is degenerate. Why? [26, exercise 3.4.5 on page 152]

9. Suppose that a planar system is transformed so that the bifurcation point is at the origin $(x, y, \mu) = (0, 0, 0)$, and so that the system with $\mu = 0$ is given by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} + \begin{pmatrix} f(x, y, 0) \\ g(x, y, 0) \end{pmatrix}$$

(a) Show that l_1 for this system is given by

$$l_1 = \frac{1}{16}\left(f_{xxx} + g_{xxy} + g_{yyy}\right)$$
$$+ \frac{1}{16\omega}\left[f_{xy}\left(f_{xx} + f_{yy}\right) - g_{xy}\left(g_{xx} + g_{yy}\right) - f_{xx}g_{xx} + f_{yy}g_{yy}\right],$$

If $l_1 < 0$ stable periodic solutions emerge from the Hopf bifurcation point as the bifurcation parameter is increased through zero whereas is $l_1 > 0$ unstable limit cycles emerge from the Hopf bifurcation point as the bifurcation parameter is decreased through zero. Furthermore, if $l_1 > 0$ the steady-state solution is unstable at the bifurcation points whereas if $l_1 < 0$ the steady-state solution is stable at the bifurcation point. [26, Theorem 3.4.2 on pages 151–152]

(b) Using the statement at the end of the previous part of this question deduce that the steady-state solution $(x, y) = (1, 1)$ of the system

$$\dot{x} = x^2 - xy,$$
$$\dot{y} = -y + x^2,$$

is stable.

Hint. You will need to transform coordinates. [26, page 153]

(c) Calculate the stability type of the degenerate steady-state solution $(x, y) = (0, 0)$ for the system

$$\dot{x} = -y + \alpha y^2 + \beta x^2 y,$$
$$\dot{y} = x - \gamma y^2 + \delta xy - y^3$$

How does the stability depend upon the values of the coefficients α, β, γ and δ? [26, Exercise 3.4.6 on page 153]

10. Show that the system $\ddot{x} + \mu\dot{x} + \nu x + x^2\dot{x} + x^3 = 0$ undergoes Hopf bifurcations on the lines $B_1\ \{\mu = 0|\ \nu > 0\}$ and $B_2\ \{\mu = \nu|\ \nu < 0\}$. Show that the former is supercritical and occurs at the steady-state solution $(0, 0)$ while the latter is subcritical and occurs simultaneously at the steady-state solution $(x, \dot{x}) = \left(\pm\sqrt{-\nu}, 0\right)$. [26, exercise 3.4.7 on page 153]

11. Suppose that a system at the critical parameter values corresponding to a Hopf bifurcation has the form

$$\dot{x} = -\omega y + \frac{1}{2}f_{xx}x^2 + f_{xy}xy + \frac{1}{2}f_{yy}y^2 + \frac{1}{6}f_{xxx}x^3 + \frac{1}{2}f_{xxy}x^2y + \frac{1}{2}f_{xyy}xy^2 + \frac{1}{6}f_{yyy}y^3 + \ldots,$$
$$\dot{y} = \omega x + \frac{1}{2}g_{xx}x^2 + g_{xy}xy + \frac{1}{2}g_{yy}y^2 + \frac{1}{6}g_{xxx}x^3 + \frac{1}{2}g_{xxy}x^2y + \frac{1}{2}g_{xyy}xy^2 + \frac{1}{6}g_{yyy}y^3 + \ldots$$

Compute $l_1(0)$ in terms of the f's and g's. [31, Chapter 3.6, question 7]]

6.10.2 Questions on the stability of limit-cycles

1. (a) Transform the equations

$$\dot{x_1} = x_2 + x_1(1 - r)(2 - r),$$
$$\dot{x_2} = -x_1 + x_2(1 - r)(2 - r),$$

into polar coordinates.

(b) Show that $r = 1$ and $r = 2$ are periodic solutions, but that, while $r = 1$ is asymptotically stable, $r = 2$ is unstable.

(c) Using linearised stability analysis show that the zero solution is unstable.

(d) Use the Liapunov function $V = \frac{1}{2}r^2$ to determine the stability of the zero solution. Compare with the solution of previous question.

[25, page 110, problems 9 and 10(b)]

6.11 Things to do

Have gone through books upto, but not including, Murray.

1. Redraw figure 6.2 to include the x-y plane and to include arrows indicating time.
2. Need to check how the Britton approach to determining subcritical/supercritical described in section 6.5 accords with what has gone before.

3. According to Farkas [18, page 155], if the right-hand side f of the system is *analytic* then the following criterion is true: the bifurcation is supercritical iff for $\mu = 0$ the equilibrium $x = 0$ is asymptotically stable. (For systems with $n > 2$ I think we require all other eigenvalues to have strictly negative real part).

   Farkas, M. (1994). *Periodic Motions.* New York: Springer.

4. Grimshaw gives the following additional references.

   Chow, S.N. and Hale, J.K. (1982). *Methods of bifurcation theory.* Springer-Verlag.

   Iooss, G., and Joseph, D.D. (1989). *Elementary stability and bifurcation theory.* Springer-Verlag.

5. Guckenheimer and Holmes comment that

   "Allwright [1977] and Mees [1981] have obtained a Hopf bifurcation criteria by means of harmonic balance and the use of a Liapunov function approach."

   Allwright, D.J. (1977). Harmonic balance and the Hopf bifurcation. *Math. Proc. Camb. Phil. Soc.*, **82**, 453–467.

   Mees, A.I. (1981). *Dynamics of Feedback Systems.* Wiley: New York.

6. Rayleigh's equation. Brusselator. Hodgkin. Huxley. Eccles. FitzHugh. Define and elaborate! (all in questions section).

A good book to read is J.E. Marsden & M. McCracken. 1976. The Hopf bifurcation and its applications Springer-Verlag.