

Chapter 8

Degenerate Hopf bifurcations

8.1 Introduction

8.1.1 Recap of the Hopf bifurcation theorem

Suppose that $\mathbf{x}(\lambda)$ is a steady-state solution of the planar system

$$\begin{aligned}\frac{dx}{dt} &= f(x, y, \lambda), \\ \frac{dy}{dt} &= g(x, y, \lambda),\end{aligned}\tag{8.1}$$

($\lambda \in \mathbb{R}$, $f \in C^2$) with eigenvalues

$$\lambda_{1,2}(\lambda) = \mu(\lambda) \pm i\omega(\lambda).$$

In chapter 6 we learnt that a Hopf bifurcation occurs at the steady-state solution $\mathbf{x}(\lambda)$ when the following conditions hold.

H(1) The eigenvalues are purely imaginary at $\lambda = \lambda_0$, i.e.

$$\begin{aligned}\mu(\lambda_0) &= 0, \\ \omega(\lambda_0) &= \omega_0 > 0.\end{aligned}$$

In addition, the non-degeneracy conditions (2 & 3) given below must be satisfied.

$$\text{H(2)} \quad \left. \frac{d}{d\lambda} \operatorname{Re} \lambda_{1,2}(\lambda) \right|_{\lambda=\lambda_0} \neq 0.$$

H(3) $l_1(\lambda_0) \neq 0$, where l_1 is known as the first Lyapunov coefficient.

(See theorem 6.1).

At a Hopf bifurcation point (x_0, y_0, λ_0) the planar system (8.1) must satisfy the following conditions

$$\begin{aligned}f(x_0, y_0, \lambda_0) &= 0, \\ g(x_0, y_0, \lambda_0) &= 0, \\ \operatorname{tr} J(x_0, y_0, \lambda_0) &= 0,\end{aligned}\tag{8.2}$$

with non-degeneracy conditions

$$\begin{aligned}\det J(x_0, y_0, \lambda_0) &> 0, \quad (\text{H1}) \\ \left. \frac{d}{d\lambda} (\operatorname{tr} J) \right|_{\lambda=\lambda_0} &\neq 0, \quad (\text{H2}) \\ l_1(\lambda_0) &\neq 0, \quad (\text{H3}).\end{aligned}$$

8.1.2 The Hopf bifurcation curve and degenerate Hopf bifurcations

Consider now a planar system containing a primary bifurcation parameter (λ) and a secondary bifurcation parameter (α)

$$\frac{dx}{dt} = f(x, \lambda, \alpha), \quad x \in \mathbb{R}^2, \quad \lambda, \alpha \in \mathbb{R},$$

and $f \in C^2$. Suppose that at the point ($\lambda = \lambda_0, \alpha = \alpha_0$) this system has a steady-state solution $x = x^0$ for which the Hopf bifurcation conditions are satisfied. System (8.2) defines a curve Γ in \mathbb{R}^4 passing through the point ($\lambda = \lambda_0, \alpha = \alpha_0$). Each point on this curve specifies a steady-state solution of equation (8.1) with $\Lambda_{1,2} = i\omega_0, \omega_0 > 0$, as long as $\det J > 0$. The standard projection of this curve onto the (λ, α) plane yields the *Hopf bifurcation curve* \mathcal{B}_H .

Let the parameters (λ, α) be varied simultaneously to track the Hopf bifurcation curve. As we are varying *two* parameters we expect to find codimension two bifurcations along the bifurcation curves. Along the co-dimension 1 bifurcation curve some of the non-degeneracy conditions can be violated¹.

Consider the Hopf bifurcation curve \mathcal{B}_H for system (8.1). At a typical point on this curve, the system has an equilibrium with a simple pair of purely imaginary eigenvalues $\lambda_{1,2} = \pm i\omega_0, \omega_0 > 0$. Furthermore, the coefficient $l_1 \neq 0$ at a non-degenerate Hopf point. While moving along the curve, one can encounter the following possibilities:

1. The condition H(1) can break down. That is the the value of the purely imaginary eigenvalue (ω_0) can approach 0 so that there are two zero eigenvalues

$$\lambda_{1,2} = 0.$$

At this point, two purely imaginary eigenvalues collide. These are the conditions for the *Bogdanov-Takens* (or *double zero eigenvalue*) bifurcation. This bifurcation can be located along a Hopf bifurcation curve or a limit point bifurcation curve. Along either curve a double-zero eigenvalue will appear as an isolated point.

For a planar system a double-zero eigenvalue occurs when the steady-state conditions (8.2) hold in addition to the conditions

$$\begin{aligned} \text{tr } J &= 0, \\ \det J &= 0. \end{aligned}$$

Sometimes this point is *not* considered to be a non-degenerate Hopf bifurcation because the non-degeneracy conditions (H2 & H3) are not violated. (It should be noted that if $\det J < 0$ then the condition $\text{tr } J = 0$ has no special significance because the eigenvalues of the Jacobian matrix J are real.)

2. The non-degeneracy condition(H2) can break down. That is the eigenvalues no longer cross the imaginary axis transversally and

$$\left. \frac{d}{d\lambda} \text{Re } \lambda_{1,2}(\lambda) \right|_{\lambda=\lambda_0} = \left. \frac{d}{d\lambda} \text{tr } J(\alpha) \right|_{\lambda=\lambda_0} = 0.$$

This bifurcation is frequently called the “double Hopf bifurcation” in the chemical engineering literature and is frequently not mentioned in text books. (Gray & Roberts [33] refer to this bifurcation as the H2₁ bifurcation because the condition H(2) is violated, with the subscript denoting the degree by which the condition has been broken).

The defining and non-degeneracy conditions for this bifurcation in a planar system are [33]

$$\begin{aligned} f = g = \text{tr } J &= \frac{d}{d\lambda} (\text{tr } J) = 0, \\ \frac{d^2}{d\lambda^2} (\text{tr } J) &\neq 0, \\ l_1 &\neq 0. \end{aligned}$$

¹In higher dimensions there is an extra possibility: extra eigenvalues can approach the imaginary axis

3. The non-degeneracy condition (H3) can break down. That is the first Lyapunov coefficient l_1 might vanish whilst $\lambda_{1,2} = \pm\omega_0 \neq 0$.

This bifurcation is often called a *generalized* (or *degenerate*) Hopf bifurcation. We follow [42] and refer to it as a *Bautin bifurcation*; see historical comments in section 8.5.2. At this point, a “soft” Hopf bifurcation turns into a “sharp” one (or vice-versa). (Gray & Roberts [33] refer to this bifurcation as the $H3_1$ bifurcation because the condition H(3) is violated, with the subscript denoting the degree by which the condition has been broken).

The Bautin bifurcation *cannot* be detected by merely monitoring the eigenvalues of the equilibrium point. One has to take into account the quadratic and cubic Taylor series coefficients of the right-hand side of (8.1) at the equilibrium.

The defining and non-degeneracy conditions for this bifurcation in a planar system are [33]

$$\begin{aligned} f = g = \operatorname{tr} J = l_1 &= 0, \\ \frac{d}{d\lambda}(\operatorname{tr} J) &\neq 0, \\ l_2 &\neq 0. \end{aligned}$$

The double-hopf bifurcation and the Bautin bifurcation are discussed further in sections 8.3 & 8.4. The double-zero bifurcation may be discussed next year...

8.2 The double-zero (Bogdanov-Takens) bifurcation

In this section we learn about a bifurcation that occurs when a steady-state solution has two zero eigenvalues. This bifurcation has several names. The most common are the ‘double-zero eigenvalue bifurcation’, which is obviously very descriptive, and the ‘Bogdanov-Takens bifurcation’, named after the two researchers who were the first to comprehensively investigate it.

Consider the planar system

$$\begin{aligned} \dot{x} &= f(x, y, \lambda, \alpha), \\ \dot{y} &= g(x, y, \lambda, \alpha). \end{aligned}$$

with smooth functions f and g and where $\alpha \in \mathbb{R}^2$. Suppose that when $\alpha = 0$ the steady-state solution $x = y = 0$ has a double-zero eigenvalue

$$\Lambda_{1,2} = 0.$$

Then, provided certain non-degeneracy conditions are satisfied, a Bogdanov-Takens bifurcation occurs, at $\alpha = 0$.

To do. Discuss the non-degeneracy conditions according to Kuznetsov.

For the two-variable system,

$$\begin{aligned} \frac{dx}{dt} &= f(x, y, \mu, \mathbf{p}), \\ \frac{dy}{dt} &= g(x, y, \mu, \mathbf{p}), \end{aligned} \tag{8.3}$$

the conditions for a Bogdanov-Takens bifurcation are given by [33, pages 382 & 383]

$$f = g = \det J = \operatorname{tr} J = 0 \tag{8.4}$$

along with the non-degeneracy conditions

$$\frac{d}{d\mu}(\operatorname{tr} J) \neq 0, \quad l_2 \neq 0, \quad \frac{d^2\mu}{dx^2} \neq 0, \quad \frac{d^2\mu}{dx^2} \neq \infty. \tag{8.5}$$

When these conditions hold the bifurcation can be interpreted as the merging of a Hopf bifurcation with a limit-point bifurcation in the unfolding diagram. If any of the non-degeneracy conditions are violated then the consequence is the formation of a degenerate Bogdanov-Takens point.

There are several (equivalent) normal forms for the double-zero eigenvalue bifurcation. These include the normal forms introduced by Bogdanov, theorem 8.1, and Takens, theorem 8.2, [42, page 279].

Theorem 8.1 (Topological normal form for the double-zero bifurcation obtained by Bogdanov) *Any generic two-parameter system*

$$\dot{x} = f(x, \mu),$$

having, at $\mu = 0$, a steady-state solution that exhibits the Bogdanov-Takens bifurcation is locally topologically equivalent near the steady-state solution to the normal form

$$\begin{aligned}\dot{\eta}_1 &= \eta_2, \\ \dot{\eta}_2 &= \beta_1 + \beta_2\eta_1 + \eta_1^2 + s\eta_1\eta_2,\end{aligned}\tag{8.6}$$

where $s = \pm 1$.

Theorem 8.2 (Topological normal form for the double-zero bifurcation obtained by Takens)

$$\begin{aligned}\dot{\eta}_1 &= \eta_2 + \beta_2\eta_1 + \eta_1^2, \\ \dot{\eta}_2 &= \beta_1 + s\eta_1^2,\end{aligned}\tag{8.7}$$

where $s = \pm 1$.

Question 8.1 *Analyse the steady-state solutions of equation (8.6) in the $\beta_1 - \beta_2$ plane for the case $s = -1$.*

1. Show that the limit-point bifurcation occurs along the curve

$$LP = \{(\beta_1, \beta_2) : 4\beta_1 - \beta_2^2 = 0\}$$

2. Show that the steady-state solutions are given by

$$E_{1,2} = (\eta_1, \eta_2) = (\eta_{1,2}^0, 0) = \left(\frac{1}{2} \left(-\beta_2 \mp \sqrt{\beta_2^2 - 4\beta_1}\right), 0\right).$$

3. Show that crossing the curve LP from right to left in the (β_1, β_2) plane implies the appearance of two steady-state solutions.
4. Show that the equilibrium E_1 undergoes a non-degenerate Hopf bifurcation along the line

$$H = \{(\beta_1, \beta_2) : \beta_1 = 0, \beta_2 < 0\}.$$

5. Show that the Hopf bifurcation gives rise to a stable limit cycle, by showing that the Lyapunov coefficient l_1 is negative ($l_1 < 0$) near the point $(\beta_1, \beta_2) = (0, 0)$.
6. Show that at the Bogdanov-Takens point $(\beta_1, \beta_2) = (0, 0)$ the steady-state solution has two zero eigenvalues.

Figure 8.1 shows the limit-point and hopf unfolding curves in the (β_1, β_2) plane. Observe that they divide the plane into three regions.

We now consider how the phase portrait of system (8.6) varies as we move around the plane in a clockwise direction starting in region one.

In region one there are no steady-state solutions. It follows that there are no limit cycles in this region.² As we enter into region 2 from region 1 by crossing the limit-point curve two steady-state solutions are created. The

²A proof of this fact should go into a latter edition of these notes

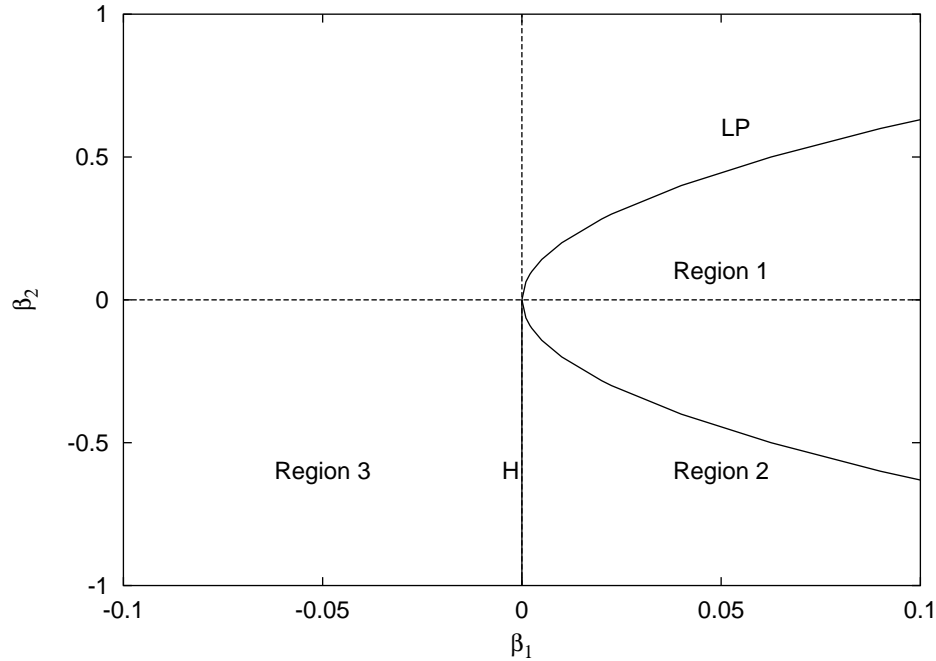


Figure 8.1: Diagram obtained by analysing system (8.6). (Question (8.1)). Nomenclature: LP and H denote the limit-point and Hopf unfolding curves respectively.

steady-state solutions can be shown to be a saddle and a stable node. As we cross the Hopf bifurcation curve H the steady-state solution E_1 , the stable node, loses stability, becoming a focus, and a stable limit cycle is present for parameter values close to the left of this curve.

Continuing the journey clockwise from region three we return to region one by crossing over the limit point curve. Two steady-state solutions are destroyed and we have no steady-state solutions. But what happened to the limit cycle curve that was created when we entered into region three? For we started by stating there region one could contain no limit cycles because there are no steady-state solutions and it appears that we have proven that there must be a limit cycle!

Thus there must be a ‘global bifurcation’ that destroys the limit cycle, that is formed when crossing the Hopf bifurcation curve. This global bifurcation must occur after the ‘H’ curve is crossed in the third quadrant and before the ‘T’ curve is crossed in the first quadrant. Furthermore, the global bifurcation curve must originate at the origin. It can be shown that this global bifurcation must be a saddle homoclinic bifurcation and that this global bifurcation has the representation [42, Lemma 8.7, page 281]

$$\text{HC} = \left\{ (\beta_1, \beta_2) : \beta_1 = -\frac{6}{25}\beta_2^2 + O(\beta_2^2), \beta_2 < 0 \right\}.$$

Furthermore, it can be shown that for $|\beta| \ll 1$, that the system (8.6) has a unique and hyperbolic stable cycle for parameter values inside the region bounded by the Hopf bifurcation curve H and the homoclinic bifurcation curve P, and no cycles outside this region.

Figure 8.2 shows the full unfolding diagram for the Bogdanov-Takens bifurcation.

To do. Provide phase-diagram portraits for the four regions.

Let us return to our clockwise journey around the unfolding diagram, this time using figure 8.2. We resume our journey as we cross the Hopf bifurcation curve, at which one of the steady-state solutions loses stability and generates a stable limit cycle. As we continue to move in a clockwise direction the cycle ‘grows’ in size and approaches the saddle point. At the homoclinic bifurcation curve HC the limit cycle touches the saddle point, becoming a homoclinic orbit. Thus as we move from region three to region four by crossing the HC the stable limit cycle is destroyed. In region four the steady-state solutions are an unstable node and a saddle.

The Bogdanov-Takens bifurcation gives rise to a *limit cycle* bifurcation, namely, the appearance of the homo-

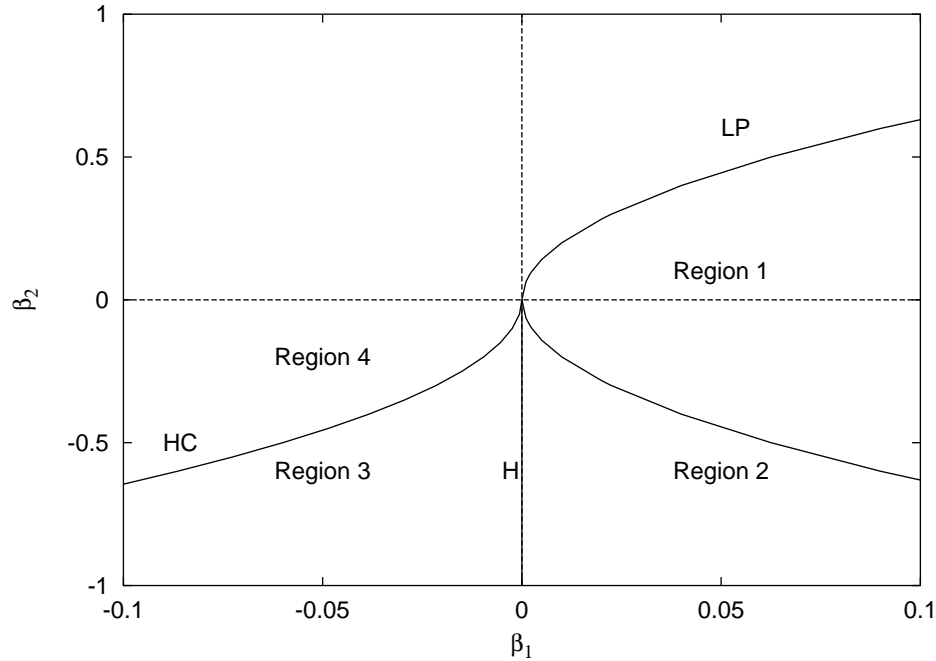


Figure 8.2: Diagram obtained by fully analysing system (8.6). Nomenclature: H, HC and LP denote the Hopf, homoclinic and limit-point unfolding curves respectively.

clinic orbit, for nearby parameter values. This bifurcation is important because it provides one of the few ways in which the homoclinic bifurcation can be detected analytically.

Question 8.2 Show that when $\beta_1 = \beta_2 = 0$ that the system (8.6) has exactly two asymptotic orbits (one tending to the steady-state solution for $t \rightarrow \infty$ and one approaching it as $t \rightarrow -\infty$). [42, page 281]

Remark 8.1 The case $s = +1$ can be treated similarly. It can be reduced to the case $s = -1$ by making the substitution $t \rightarrow -t$, $\eta_2 \rightarrow -\eta_2$. It follows that the unfolding diagram remains the same with the exception that the cycle becomes unstable near the Bogdanov-Takens point. [42, page 281]

Example 8.1 Consider the system [33]

$$\begin{aligned}\frac{dx}{dt} &= x^2 + \mu y + \beta_1, \\ \frac{dy}{dt} &= \alpha x + \lambda^2 - y.\end{aligned}$$

Obtain the condition for the Bogdanov-Takens bifurcation to occur and plot this curve in the $\alpha - \beta$ plane.

8.3 The double Hopf bifurcation

The defining and non-degeneracy conditions for the double Hopf bifurcation in a planar system are [33]

$$\begin{aligned}f = g = \text{tr } J = \frac{d}{d\lambda} (\text{tr } J) &= 0, \\ \frac{d^2}{d\lambda^2} (\text{tr } J) &\neq 0, \\ l_1 &\neq 0.\end{aligned}\tag{8.8}$$

There are two varieties of the double Hopf bifurcation depending upon the values of l_1 and $\frac{d^2}{d\lambda^2}(\text{tr } J)$. This year I will describe the most common one.

Consider a planar system which can be written in the polar form

$$\begin{aligned}\dot{\rho} &= -\rho(\rho^2 + \lambda^2 - \alpha) \\ \dot{\theta} &= 1,\end{aligned}\tag{8.9}$$

Figure 8.3 shows the two generic steady-state diagrams that this system as a function of the secondary bifurcation parameter (α). At $\alpha = 1$, figure 8.3 (a), the steady-state diagram contains two Hopf bifurcation points and a branch of limit cycles connects the two Hopf bifurcation points. At $\alpha = -1$, figure 8.3 (b), the steady-state diagram no longer contains Hopf bifurcation points.

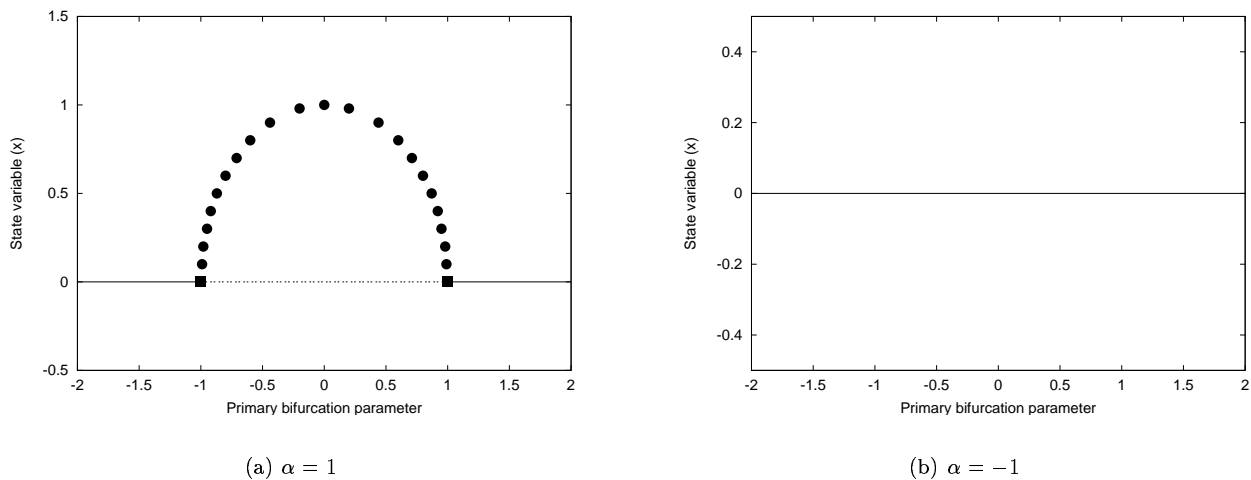


Figure 8.3: Generic steady-state diagrams for system (8.9). Between the values $\alpha = 1$, diagram (a), and $\alpha = 0$, diagram (b), a double-hopf bifurcation occurs.

Question 8.3 Find the steady-state solutions of system (8.9) and determine their stability as a function of the primary bifurcation parameter, λ , and the secondary bifurcation parameter, α . Hence account for the steady-state diagrams shown in figure 8.3.

As the value of the secondary bifurcation parameter, α , is changed the location of the Hopf bifurcation points, should they exist, also changes. Figure 8.4 shows the values of the primary bifurcation parameters at which a Hopf bifurcation occurs as a function of the secondary bifurcation parameter. A diagram that shows how the value of the primary bifurcation parameter, at which a particular type of bifurcation occurs, varies as a function of a secondary bifurcation diagram is known as an unfolding diagram. The unfolding diagram shows that α is reduced towards zero, the two values of λ at which Hopf bifurcation points occur become closer together. At $\alpha = 0$ the two Hopf bifurcation points come together at a $H2_1$ degeneracy. For values of the secondary bifurcation parameter below zero there are no Hopf bifurcation points.

Question 8.4 Figure 8.4 shows the Hopf bifurcation point locus for system 8.9.

1. Write down the parametric form of the equation plotted in figure 8.4.
2. Show that the $H2_1$ degeneracy occurs at the point $(x, \lambda, \alpha) = (0, 0, 0)$.

Hint. We can not find the location of the $H2_1$ point by applying conditions 8.8 as the governing equation use polar coordinates rather than cartesian coordinates. Instead, think of figure 8.4 as a steady-state diagram for a singularity function G . What kind of bifurcation occurs on this figure?

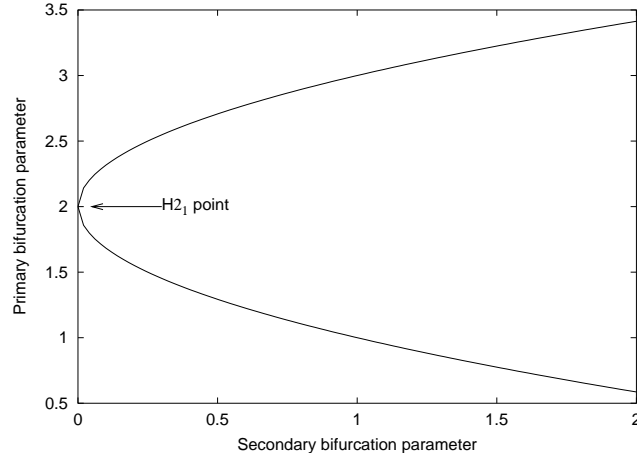


Figure 8.4: Unfolding diagram for the $H2_1$ degeneracy (type A). The curve is the locus of Hopf points.

8.4 The Bautin (generalized Hopf) bifurcation

Consider the planar system

$$\begin{aligned}\dot{x} &= f(x, y, \lambda, \alpha), \\ \dot{y} &= g(x, y, \lambda, \alpha).\end{aligned}$$

with smooth functions f and g . Suppose that $x = y = 0$ is a steady-state solution with eigenvalues

$$\Lambda_{1,2} = \mu(\lambda) \pm i\omega(\lambda)$$

The defining and non-degeneracy conditions for a Bautin (generalized Hopf) bifurcation for this system are [42]

$$\begin{aligned}f = g = \text{tr } J = l_1 &= 0, \\ \frac{d}{d\lambda}(\text{tr } J) &\neq 0, \\ \det J &> 0, \\ l_2 &\neq 0,\end{aligned}\tag{8.10}$$

where the function l_2 is called the second Liapunov coefficient. We further require that at the Hopf bifurcation point (x_0, y_0, α_0) that

$$\begin{vmatrix} \frac{\partial \text{tr } J}{\partial \lambda} & \frac{\partial \text{tr } J}{\partial \alpha} \\ \frac{\partial l_1}{\partial \lambda} & \frac{\partial l_1}{\partial \alpha} \end{vmatrix}_{\alpha=\alpha_0} \neq 0.\tag{8.11}$$

These conditions ensure that at the Hopf bifurcation point the steady-state solutions has purely imaginary eigenvalues, $\lambda_{1,2} = \pm i\omega_0, \omega_0 > 0$. Note that in order to have calculated l_1 we may have had to use a coordinate transformation to fix the steady-state solution at the origin.

It can be shown that the normal form of the Bautin bifurcation is given in polar coordinates by [42, page 270]

$$\begin{aligned}\dot{\rho} &= \rho(\beta_1 + \beta_2\rho^2 - \rho^4), \\ \dot{\theta} &= 1.\end{aligned}\tag{8.12}$$

In these equations the first liapunov coefficient is given by β_2 .

The equations in (8.12) are independent. The second equation describes a rotation with unit angular velocity. The trivial equilibrium $\rho = 0$ of the first equation corresponds to an equilibrium, $x = y = 0$, of the planar system. Positive equilibria of the first equation in (8.12) satisfy

$$\beta_1 + \beta_2\rho^2 - \rho^4 = 0\tag{8.13}$$

and describe circular limit cycles. Equation (8.13) can have zero, one, or two positive solutions (cycles).

Question 8.5 Analyse the steady-state solutions of equation (8.13) in the $\beta_1 - \beta_2$ plane.

1. Identify parameter regions in which there are zero, one and two limit cycles.
2. Determine the stability of the limit-cycles identified in your answer to the previous question.

8.5 Conclusions

8.5.1 Summary

8.5.2 Historical remarks

Generic two-parameter bifurcation diagrams near a point where the first Liapunov coefficient $l_1(0)$ vanish were first obtained by Bautin [12]. Therefore, we follow Kuznetsov and call this bifurcation the *Bautin bifurcation*. Serebryakova [63] computer the second Liapunov coefficient $l_2(0)$ in terms of real Taylor coefficients of a general system and derived asymptotic formulas for the cyclic fold-bifurcation curve near the Bautin point. Takens [68] has studied higher degeneracies at the Hopf bifurcation with $\text{codim} > 2$ (for example, the case $l_2(0) = 0$).

The main features of the bifurcation diagram near an equilibrium with a double zero eigenvalue were known to mathematicians of the Andronov school in the late 1960s and were routinely used in the analysis of concrete models, (see, for example, [13, pages 183–186]). However, the complete picture, including the uniqueness of the limit cycle is due to Bogdanov [19] (his results were announced by Arnold in 1972) and Takens [69].

The above is from [42, page 345].

8.6 Maple commands

8.7 Revision of key ideas

The following questions are about the key ideas in this chapter.

8.8 Questions on degenerate Hopf bifurcation

8.8.1 The Bautin bifurcation

1. The following predator-prey system is a generalisation of the Volterra equations [16]

$$\begin{aligned}\dot{x} &= \frac{x^2(1-x)}{n+x} - xy, \\ \dot{y} &= -\gamma y(m-x),\end{aligned}$$

where m, n , and γ are positive parameters. (See also [14] for more information about this model).

- (a) Derive the equation for the Hopf bifurcation curve in the system and show that it is independent of γ .
- (b) Compute the first Liapunov coefficient along the Hopf curve and show that it vanishes at a Bautin point when

$$(m, n) = \left(\frac{1}{4}, \frac{1}{8}\right).$$

- (c) Compute the second Liapunov coefficient and prove that the Bautin bifurcation is non-degenerate for all $\gamma > 0$. (Not covered in the lecture notes yet).
- (d) Sketch the parametric portrait on the (m, n) -plane and all possible phase portraits of the system. Hint: see [14, page 42].

[42, exercise 3 on page 324]

2. A model for a laser with a controllable resonator is given by [60] is given by

$$\begin{aligned}\dot{m} &= Gm \left(n - \frac{\beta}{\rho m + 1} - 1 \right), \\ \dot{n} &= \alpha - (m + 1)n,\end{aligned}$$

where $m \geq 0$, $\rho > 0$, $G > 1$, and $\alpha, \beta > 0$.

Show that this model has a Bautin bifurcation on the Hopf curve on the (α, β) plane if

$$\rho < 1,$$

and

$$\rho - 1 + G\rho > 0.$$

(Hint. Parametrise the Hopf bifurcation curve by the m -coordinate of the nontrivial equilibrium. See [13, pages 430-329].)

8.8.2 The double-zero eigenvalue

(a) Show that the averaged forced Van der Pol oscillator [37]

$$\begin{aligned}\dot{x}_1 &= \alpha_1 x_2 + x_1 (1 - x_1^2 - x_2^2), \\ \dot{x}_2 &= \alpha_1 x_1 + x_2 (1 - x_1^2 - x_2^2) - \alpha_2\end{aligned}$$

exhibits a non-degenerate Bogdanov-Takens bifurcation at $(\alpha_1, \alpha_2) = \left(\frac{1}{2}, \frac{1}{2}\right)$. (Hint. The critical steady-state solution with a double zero eigenvalue has coordinates $(x_1, x_2) = \left(\frac{1}{2}, \frac{1}{2}\right)$.) [42, Exercise 7a in section 8.7]

(b) Show that a prototype reference adaptive control system with the σ -modification adaptation law [62]

$$\begin{aligned}\dot{x}_1 &= x_1 - x_1 x_2 + 1, \\ \dot{x}_2 &= \alpha_1 x_2 + \alpha_2 x_1^2,\end{aligned}$$

has a non-degenerate Bogdanov-Takens bifurcation at $\alpha = \left(-\frac{2}{3}, \frac{8}{81}\right)$. Sketch the phase portraits of the system near the BT-point. [42, Exercise 7a in section 8.7]

(c) Consider the predator-prey model [15]

$$\begin{aligned}\dot{x}_1 &= x_1 - \frac{x_1 x_2}{1 + \alpha x_1}, \\ \dot{x}_2 &= -\gamma x_2 + \frac{x_1 x_2}{1 + \alpha x_1} - \delta x_2^2.\end{aligned}$$

Show that this has Hopf and fold bifurcation curves in the (α, δ) -plane that touch at a Bogdanov-Takens point. Prove, at least for $\gamma = 1$, that this codim 2 bifurcation is non-degenerate. [42, Exercise 8 in section 8.7]

8.9 Things to do

1. Answer question 8.5 in the text.
2. Read Takens [68]

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