Chapter 3

Second-order differential equations: Steady-state solutions and their stability

3.1 Introduction

In this chapter we consider systems comprising two autonomous first-order ordinary differential equations (ODEs). Such a system can be written in the form

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= f\left(x, y\right), \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= g\left(x, y\right), \end{aligned} \tag{3.1}$$

or in vector form

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = f\left(\mathbf{x}\right), \quad \mathbf{x} \in \mathbb{R}^2. \tag{3.2}$$

The system (3.1) is often called a planar dynamical system or planar system.

Many mathematical models can be written in the form of (3.1) and for such models we want to analyse their behaviour. Of course, we can always investigate (3.1) by numerical simulation. Indeed integration of (3.1), provided that initial conditions are also specified, is very easy using standard routines in mathematical packages such as maple, matlab and mathematica. However, to fully understand the behaviour of system (3.1) we might have to compute many numerical solutions.

In chapter 1 we saw that the behaviour of a single differential equation can be predicted without solving the equation numerically. Similarly although computer simulations can be a useful aid, much insight into the behaviour of (3.1) can be obtained, without integrating the system numerically, by performing some 'routine' calculations: identify the steady-state solutions and determine their stability.

Steady-state solutions of (3.1) are found by putting the time-derivatives equal to zero and solving the resulting set of equations. This is described in section 3.2. The key tool to determine stability is the technique of *linearised stability* which is described in section 3.3. This is a generalisation of the method used in chapter 1.4 to determine the stability of a steady-state solution for a scalar differential equation. The method of linearised stability involves approximating system (3.1) by a linear system. This method gives *local* information, i.e. it describes what happens when a system is perturbed by a 'small' amount away from a steady-state solution. There are occasions when this analysis fails to give useful information. (For the scalar case considered in chapter 1.4 this happened when the eigenvalue was zero). An alternative approach to determine stability, described in section 3.5, is to to use Liapunov's direct method. This is a *global method* that determines the stability of a steady-state solution of (3.1) directly, i.e. the system is not approximated by a linear system.

Finally, we note that a second-order ordinary differential equation can be rewritten as a system of two first-order differential equations.

Example 3.1 *Write the equation*

$$m\frac{d^2x}{dt^2} + k\left(x + x^3\right) = 0,$$

as a system of two first-order differential equations [33, page 5].

(This equation models the motion of a particle of mass m attached to a spring of stiffness $k\left(x + x^3\right)$, $k > 0$, where x is displacement.

Solution To convert a second-order ODE to a system of two first-order ODEs start by letting $\frac{\mathrm{d}x}{\mathrm{d}t} = y$. Then we have

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= y,\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\frac{k}{m}\left(x + x^3\right).\end{aligned}$$

□

3.2 Steady-state solutions

Steady-state solutions (x^*, y^*) of system (3.1) are found by solving the system of equations

$$\begin{aligned}f\left(x^*, y^*\right) &= 0,\\ g\left(x^*, y^*\right) &= 0.\end{aligned}$$

Finding the steady-state solutions is a matter of algebraic manipulation, the ease or difficulty of which is specific to the model at hand.

Example 3.2 *Find the steady-state solutions of the system*

$$\begin{aligned}\frac{dx}{dt} &= x - y^3,\\ \frac{dy}{dt} &= 7x - 7y.\end{aligned}$$

Solution Putting the time derivatives equal to zero we find that we have to solve the system of equations

$$\begin{aligned}x^* - {y^*}^3 &= 0,\\ 7x^* - 7y^* &= 0.\end{aligned}$$

During the calculation to find the steady-state solutions I will drop the superscript notation. The second of these equations implies that

$$x = y.$$

The first equation therefore becomes

$$\begin{aligned}x - x^3 &= 0,\\ x\left(1 - x^2\right) &= 0.\end{aligned}$$

Thus the steady-state values for x are $x_1 = 0$, $x_2 = 1$ and $x_3 = -1$. The corresponding values for y are $y_1 = 0$, $y_2 = 1$ and $y_3 = -1$. The steady-state solutions are therefore, going back to the superscript notation,

$$\begin{aligned}(x_1^*, y_1^*) &= (0,0)\,,\\ (x_2^*, y_2^*) &= (1,1)\,,\\ (x_3^*, y_3^*) &= (-1,-1)\,.\end{aligned}$$

□

Definition 3.1 (Trivial solution) *The steady-state solution* $(x^*, y^*) = (0,0)$*, in vector form* $\mathbf{x} = \mathbf{0}$*, is called the trivial solution.*

Note that in many planar systems used in mathematical modelling we will only be interested in steady-state solutions in which both components of the solution are positive, i.e. $x^* > 0$ and $y^* > 0$. For example, in problems from mathematical ecology, modelling competing populations, or problems from chemistry, modelling the concentrations of chemicals, the components of the steady-state solution have no physical meaning if they are negative.

After a steady-state solution (x^*, y^*) has been found, the next task is to investigate its stability

3.3 Linearised stability of steady-state solutions

In this section we generalise the method used in chapter 1.4 to determine the stability of a steady-state solution of a single differential equation to find the stability of a steady-state solution of (3.1). As in chapter 1.4 the method involves deriving a linear approximation to our, in general non-linear, model. This model is used to determine what happens if a small perturbation is applied to the dependent variables, moving them away from a steady-state solution.

In section 3.3.1 we derive the linearised approximation. This is solved in section 3.3.2, where we derive conditions for a steady-state solution to be stable or unstable. In chapter 1.4 the method of linearised stability did not reveal the stability when the eigenvalue was equal to zero. Similarly, there are circumstances for planar systems in which linearised stability does not determine the stability of a steady-state solution. This is discussed in section 3.3.3.

In section 3.5 we discuss how the stability/instability of a steady-state solution can be determined using Liapunov's direct method. In principle this method can be used to determine the stability of a steady-state solution in circumstances under which linearised stability fails. Furthermore, as a by-product of proving stability other useful information may be obtained.

3.3.1 Derivation of the linearised system

In chapter 1.4.4 we showed how the stability of a steady-state solution, $x = x^*$, for a single differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x)\,,$$

can be investigated by using a Taylor series expansion of the function $f(x)$ near the steady-state solution $x = x^*$. The same idea is used to investigate the stability of a steady-state solution of a system of two, or more, differential equations.

Suppose that (x^*, y^*) is a steady-state solution of system (3.1). Defining

$$\begin{aligned}u(t) &= x(t) - x^*,\\ v(t) &= y(t) - y^*,\end{aligned}$$

we proceed in an analogous manner to chapter 1.4.4.

1. Write the functions $f(x,y)$ and $g(x,y)$ in terms of u and v.
2. Expand the functions $f(u,v)$ and $g(u,v)$ in terms of ascending powers of u and v. To do this we use a Taylor series expansion for a function of *two* variables. (See appendix B.3 for details).

Doing this we obtain the approximate (linearised) equations

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}t} &= f_x(x^*,y^*)\,u + f_y(x^*,y^*)\,v,\\ \frac{\mathrm{d}v}{\mathrm{d}t} &= g_x(x^*,y^*)\,u + g_y(x^*,y^*)\,v.\end{aligned} \tag{3.3}$$

In obtaining the system 3.3 we have assumed that u and v are sufficiently small that we can neglect second-order terms in the Taylor series expansions of the functions f and g, i.e. expressions involving v^2, u^2 and uv.

System (3.3) can be written in vector form as

$$\dot{\mathbf{x}} = J\mathbf{x}$$

where the vector $\mathbf{x}$ is defined by

$$\mathbf{x} = \begin{pmatrix} u \\ v \end{pmatrix}$$

and the matrix J is given by

$$J = \begin{pmatrix} f_x(x^*,y^*) & f_y(x^*,y^*) \\ g_x(x^*,y^*) & g_y(x^*,y^*) \end{pmatrix}.$$

Remark 3.1 *The matrix J*

$$J(x,y) = \begin{pmatrix} f_x(x,y) & f_y(x,y) \\ g_x(x,y) & g_y(x,y) \end{pmatrix},$$

is known as the Jacobian matrix. The matrix

$$\begin{aligned} J(x^*,y^*) &= \left.\begin{pmatrix} f_x(x,y) & f_y(x,y) \\ g_x(x,y) & g_y(x,y) \end{pmatrix}\right|_{(x,y)=(x^*,y^*)},\\ &= \begin{pmatrix} f_x(x^*,y^*) & f_y(x^*,y^*) \\ g_x(x^*,y^*) & g_y(x^*,y^*) \end{pmatrix}, \end{aligned} \tag{3.4}$$

is the Jacobian matrix evaluated at the steady-state solution (x^*,y^*).

Question 3.1 *Write out a detailed derivation of the linear linear system* (3.3) *from the nonlinear system* (3.1) *by putting in the steps that were included for the case of a single differential equation, in section 1.4.4, but which were omitted the sketch argument above.*

3.3.2 Stability of the linearised system

The solution of the linear system of equations

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}t} &= f_x(x^*,y^*)\,u + f_y(x^*,y^*)\,v,\\ \frac{\mathrm{d}v}{\mathrm{d}t} &= g_g(x^*,y^*)\,u + g_y(x^*,y^*)\,v.\end{aligned}$$

is given by

$$\begin{aligned} u &= c_1 \exp[\lambda_1 t] + c_2 \exp[\lambda_2 t],\\ v &= c_3 \exp[\lambda_1 t] + c_4 \exp[\lambda_2 t]. \end{aligned} \tag{3.5}$$

In the solution the parameters c_i $(i = 1, \cdots 4)$ are constants. Equation (3.5) is a solution of the system (3.3) only if the parameters λ_j $(j = 1, 2)$ are the *eigenvalues* of the *Jacobian matrix* evaluated at the steady-state solution (x^*, y^*), equation (3.4). In most circumstances, and almost all practical circumstances, the eigenvalues determine the stability of the steady-state solution.

(In writing the solution 3.5, I have assumed, for simplicity, that the eigenvalues are distinct, $\lambda_1 \neq \lambda_2$. This does not effect what follows.) When we had a single differential equation the eigenvalue had to be real. For a system of two differential equations the eigenvalues may either be real or complex.

When is a steady-state solution stable? Recall that the stability of a steady-state solution is determined by the answer to the following question. Suppose that the system is 'at rest' at a steady-state solution, so that both time-derivatives are equal to zero, and that a 'small' perturbation is made to the initial condition. Will the solution of the differential equation, i.e. the values of $x(t)$ and $y(t)$, tend towards (stable) or away (unstable) from the steady-state?

The linearised system approaches the steady-state solution as time approaches infinity if the distances

$$u(t) = x(t) - x^*,$$
$$v(t) = y(t) - y^*,$$

both approach zero as time increases to infinity, i.e.

$$\lim_{t\to\infty} u(t) = 0,$$
$$\lim_{t\to\infty} v(t) = 0.$$

From the solution given in (3.5) we see that this only happens if the real part of both of the eigenvalues (λ_1, λ_2) are less than zero.

Question 3.2 *Explain the observation that*

$$\lim_{t\to\infty} u(t) = 0,$$
$$\lim_{t\to\infty} v(t) = 0.$$

if the real part of the eigenvalues (λ_1, λ_2) is less than zero. Consider the cases:

1. *The eigenvalues are real.*
2. *The eigenvalues are complex.*

Definition 3.2 (Stable and unstable steady-state) *A steady-state* (x^*, y^*) *of* (3.1) *is:*

stable *if the real part of both eigenvalues (λ_1, λ_2) are negative;*

unstable *if the real part of at least eigenvalue is positive.*

Remark 3.2 (Stability is undetermined) *There are two cases in which linearised stability does not determine the stability of a steady-state solution. These are:*

1. *there is one negative eigenvalue and one eigenvalue equal to zero;*
2. *there are two eigenvalues with zero real part.*

The eigenvalues, λ, of the Jacobian defined by equation (3.4) are given by

$$|J - \lambda I| = 0 \tag{3.6}$$
$$\Rightarrow \lambda^2 - (\operatorname{tr} J)\,\lambda + \det J = 0, \tag{3.7}$$

where $\operatorname{tr} J$ is the *trace* of the Jacobian matrix ($\operatorname{tr} J = f_x + g_y$) and $\det J$ is the *determinant* of the Jacobian matrix ($\det J = f_x g_y - f_y g_x$).

As noted in definition 3.2 the steady-state solution (x^*, y^*) is *stable* if the real part of each eigenvalues be less than zero, i.e.

$$\operatorname{Re} \lambda < 0.$$

The following observation is often useful. It enables the stability of a steady-state solution to be determined directly from the Jacobian matrix, without having to find its eigenvalues.

Remark 3.3 *A steady-state solution* $\bar{\mathbf{x}}$ *is stable if and only if*

$$\operatorname{tr} J < 0 \quad \textit{and} \quad \det J > 0. \tag{3.8}$$

A steady-state solution is unstable if either $\operatorname{tr} J > 0$ *or if* $\det J < 0$.

Question 3.3

1. *Derive equation* (3.7) *from equation* (3.6).
2. *Derive the stability equation* (3.8) *from equation* (3.7).

(Put this into the notes next year).

Example 3.3 *In example 3.2 we showed that the steady-state solutions of the system*

$$\frac{dx}{dt} = x - y^3,$$
$$\frac{dy}{dt} = 7x - 7y,$$

are

$$(x_1^*, y_1^*) = (0, 0)\,,$$
$$(x_2^*, y_2^*) = (1, 1)\,,$$
$$(x_3^*, y_3^*) = (-1, -1)\,.$$

Determine the stability of these steady-state solutions.

Solution The Jacobian matrix is given by

$$J = \begin{pmatrix} 1 & -3y^2 \\ 7 & 7 \end{pmatrix}.$$

At the steady-state $(x_1^*, y_1^*) = (0, 0)$ we have

$$J\,(0,0) = \begin{pmatrix} 1 & 0 \\ 7 & -7 \end{pmatrix},$$
$$\operatorname{tr} J = -6 < 0,$$
$$\det J = -7 < 0.$$

Thus the steady-state $(x_1^*, y_1^*) = (0, 0)$ is unstable.

At the steady-state $(x_2^*, y_2^*) = (1, 1)$ we have

$$J(1,1) = \begin{pmatrix} 1 & -3 \\ 7 & -7 \end{pmatrix},$$
$$\operatorname{tr} J = -6 < 0,$$
$$\det J = +14 > 0.$$

Thus the steady-state $(x_2^*, y_2^*) = (1, 1)$ is stable.

At the steady-state $(x_3^*, y_3^*) = (-1, -1)$ we have

$$J(-1,-1) = \begin{pmatrix} 1 & -3 \\ 7 & -7 \end{pmatrix},$$
$$\operatorname{tr} J = -6 < 0,$$
$$\det J = +14 > 0.$$

Thus the steady-state $(x_3^*, y_3^*) = (-1, -1)$ is stable. □

This question shows how useful 3.8 is: stability is determined without having to find the eigenvalues.

Question 3.4 *Find the eigenvalues for the three steady-state solutions in question 3.3.*

The final idea of this section is the concept of a hyperbolic steady-state solution. This concept is not important for the purposes of this course. We mention it because it is mentioned in textbooks describing the *theory* of dynamical systems.

Consider the planar differential system defined by

$$\dot{\mathbf{x}} = f(\mathbf{x}),$$

where f is smooth. Let $\bar{\mathbf{x}}$ be a steady-state solution of the system (i.e., $f(\bar{\mathbf{x}}) = \mathbf{0}$ and let J denote the Jacobian matrix evaluated at $\mathbf{x}$. Let n_-, n_0 and n_+ be the number of eigenvalues of J (counting multiplicities) with negative, zero, and positive real part, respectively.

Definition 3.3 (Hyperbolic steady-state) *A steady-state solution is called* ***hyperbolic*** *if $n_0 = 0$, that is, if there are no eigenvalues on the imaginary axis. A hyperbolic steady-state is called a* ***hyperbolic saddle*** *if $n_- n_+ \neq 0$.*

Since a generic matrix has no eigenvalues on the imaginary axis $n_0 = 0$, hyperbolicity is a typical property and an equilibrium in a generic system (i.e., one not satisfying certain special conditions) is hyperbolic. These observations can be formalised rigorously.

3.3.3 What happens when the largest eigenvalue is zero?

We know that a steady-state solution is stable when both eigenvalues have negative real parts and is unstable should at least one eigenvalue have positive real part (definition 3.2). We also know that there are two circumstances in which stability is not determined by the eigenvalues of the steady-state solution (remark 3.2): there is one negative eigenvalue and one eigenvalue equal to zero; there are two zero eigenvalues. In the latter case the steady-state solution is non-hyperbolic.

When either of the conditions noted under remark 3.2 holds then the steady-state may be stable or unstable. In these circumstances stability can can be determined using linearised stability analysis and an alternative method must be used to determine stability.

Question 3.5 *In applications we are not particularly concerned with the stability of a steady-state is undetermined by linear stability, i.e. the conditions of remark 3.2 hold. Why?*

One method that may, in principle, be used to determine the stability of a steady-state solution when linearised stability fails is Liapunov's direct method. This is described in section 3.5. We end this chapter by using an alternative method, a change of co-ordinate system, to show that the trivial solution of the system

$$\begin{aligned} \dot{x} &= y + kx\left(x^2 + y^2\right), \\ \dot{y} &= -x + ky\left(x^2 + y^2\right), \end{aligned} \tag{3.9}$$

is stable when $k < 0$ and unstable when $k > 0$[1].

Question 3.6 *Determine the eigenvalues for the trivial solution of system 3.9.*

Example 3.4

1. *Show that by defining*

$$\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta. \end{aligned}$$

(polar coordinates) system 3.9 can be written as

$$\begin{aligned} \dot{r} &= kr^3, \\ \dot{\theta} &= -1. \end{aligned}$$

Hint. Read appendix D.

2. *Deduce that the zero solution is asymptotically stable if $k < 0$ and unstable if $k > 0$.*

Solution Add the solution next year.

Remark 3.4 *A change to polar coordinates is often instructive when equation* (3.1) *contains terms of the form* $x^2 + y^2$.

3.3.4 Questions on stability of steady-state solutions

1. Determine the stability of the following linear systems [32, page 108, problem 1]

$$\begin{aligned} &(a) \quad \dot{z_1} = z_1 + 6z_2, && \dot{z_2} = -z_1 - 4z_2 \\ &(b) \quad \dot{z_1} = -7z_1 + 10z_2, && \dot{z_2} = -4z_1 + 5z_2 \\ &(c) \quad \dot{z_1} = -5z_1 + 6z_2, && \dot{z_2} = -3z_1 + 4z_2. \end{aligned}$$

2. Determine the stability of the trivial solution of the following linear systems [32, page 108, problem 2]

$$\begin{aligned} &(a) \quad \dot{z_1} = z_1 + 6z_2 - z_1^2, && \dot{z_2} = -z_1 - 4z_2 + z_1 z_2 \\ &(b) \quad \dot{z_1} = -7z_1 + 10z_2 + z_2\sin z_1, && \dot{z_2} = -4z_1 + 5z_2\cos z_1 \\ &(c) \quad \dot{z_1} = -5z_1 + 6z_2 + z_2^2, && \dot{z_2} = -3z_1 + 4z_2 - z_2^4. \end{aligned}$$

[1] This is a common example, appearing, for example, in [32, page 109–110, problems 7 & 10(a)] and [37, Example 10.7 on page 280].

3. The motion of a simple pendulum with linear damping is governed by the equation

$$u'' + \nu u' + g \sin u = 0,$$

where g and ν are positive constants. Show that the solution $u = 0$ is stable, but that the solution $u = \pi$ is unstable. [32, page 109, problem 5]

4. Find the steady-state solutions and determine their stability for the following systems. [41, pages 53-55]

$$\begin{array}{lll} \text{(a)} & \dot{x} = x, & \dot{y} = x^2 + y^2 - 1. \\ \text{(b)} & \dot{x} = y, & \dot{y} = x\left(1 - x^2\right) + y. \\ \text{(c)} & \dot{x} = x\left(1 - \dfrac{x}{2} - y\right) & \dot{y} = y\left(x - 1 - \dfrac{y}{2}\right). \end{array}$$

5. Find the steady-state solutions and determine their stability for the system

$$\dot{x} = x\left(1 - \frac{x}{7}\right) - \frac{6xy}{7 + 7x},$$
$$\dot{y} = 0.2y\left(1 - \frac{Ny}{x}\right).$$

Consider the cases $N = 0.5$ and $N = 2.5$. [41, page 70]

(This system is an example of the *Holling-Tanner* model).

6. A particular form of the *Holling-Tanner* model is given by [41, page 70]

$$\dot{x} = f\left(x, y\right) = x\left(1 - \frac{x}{7}\right) - \frac{6xy}{7 + 7x},$$
$$\dot{y} = g\left(x, y\right) = 0.2y\left(1 - \frac{Ny}{x}\right).$$

We investigate the steady-state solutions of this system, and their stability, as a function of the parameter N ($N > 0$).

(a) What does the Holling-Tanner model model?

(b) In this question we find the physically meaningful steady-state solutions of the model.

   (i) Show that there are three steady-state solutions: $\bar{\mathbf{x}}_1$, in which the species y becomes extinct and $\bar{\mathbf{x}}_\pm$.

   (ii) Show that the steady-state solution $\bar{\mathbf{x}}_-$ is not of physical interest because both components of the steady-state solution are negative.

(c) Determine the stability of the steady-state solution $\bar{\mathbf{x}}_1$.

(d) In this question we investigate the stability of the steady-state solution $\bar{\mathbf{x}}_+$ as a function of the parameter N.

   (a) Show that along this solution branch we have

$$f_x\left(\bar{\mathbf{x}}_+\right) = \frac{2x^*\left(3 - x^*\right)}{7\left(1 + x^*\right)}.$$

   (b) Show that

$$\det J = \frac{0.4x^*}{7\left(1 + x^*\right)} \cdot \frac{1}{N}\left[Nx^* + 3\left(1 - N^*\right)\right].$$

   Hence, or otherwise, deduce that

$$\det J > 0 \, \forall N > 0.$$

(c) Show that

$$\operatorname{tr} J = \frac{2x^* (3^*)}{7(1+x^*)} - 0.2,$$
$$\operatorname{tr} J = 0 \Rightarrow {x^*}^2 - 2.3x^* + 0.7 = 0.$$

(d) Hence, or otherwise, deduce that there are critical values of N, N_1 and N_2, such that the steady-state solution $\bar{\mathbf{x}}_+$ is
stable if $0 < N < N_1$
unstable if $N_1 < N < N_2$
stable if $N_2 > N$

(e) Draw steady-state diagrams showing how the values for x and y vary as a function of N.

7. Consider the system [32, page 271]

$$x' = x(\mu - x) - (x+1)y^2,$$
$$y' = y(x-1).$$

(a) Show that the steady-state solutions of this system are given by

$$(x_1, y_1) = (0, 0),$$
$$(x_2, y_2) = (\mu, 0),$$
$$(x_{3,4}, y_{3,4}) = (1, y_\pm)$$

where $y_\pm$ is given by

$$y_\pm^2 = \frac{1}{2}(\mu - 1).$$

(b) Show that the solution branch (x_1, y_1) is stable for $\mu < 0$ and unstable for $\mu > 0$.

(c) Show that the solution branch (x_2, y_2) is only stable for $0 < \mu < 1$.

(d) (i) Show that the solution branch $(x_{3,4}, y_\pm)$ is stable provided that

$$\mu - 2 - y_\pm^2 < 0.$$

(ii) Using the fact that along the solution branch we have

$$y_\pm^2 = \frac{1}{2}(\mu - 1)$$

show that the solution branch $(x_{3,4}, y_\pm)$ is stable provided that

$$y_{pm}^2 - 1 < 0$$

and deduce that the solution branch is stable for $1 < \mu < 3$.

3.4 Steady-state diagrams and bifurcation points

3.4.1 Drawing a steady-state diagram for a planar system

For a single first-order differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, \mu),$$

the steady-state diagram the steady-state diagram shows the steady-state solution(s), and their stability, as a function of the bifurcation parameter (μ). A point (μ_0, x_0) is a bifurcation point if the number of steady-state solutions in the neighbourhood of this point is not constant for any arbitrary small change in the value of the bifurcation parameter. Loosely speaking, at a bifurcation point two solution branches intersect with distinct tangents.

Given a system of two first-order differential equations

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= f\left(x, y, \mu\right),\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= g\left(x, y, \mu\right),\end{aligned}$$

how do we draw the corresponding steady-state diagram? One approach would be to draw a three-dimensional figure. One axis would be the value of the bifurcation parameter whilst the other two axis would show how the x- and y-values of the steady-state solution depend upon the value of the bifurcation parameter. There are three disadvantages with this approach. Firstly, many people have difficulty in understanding three-dimensional figures. Secondly, even using a graphics package, it is difficult to draw a good three-dimensional figure. Finally, this approach would not generalise to problems involving three, or more, first-order differential equations.

The solution is to draw two two-dimensional steady-state diagrams: one showing the x-values of the steady-state solutions as a function of the bifurcation parameter and the other showing the y-values of the steady-state solutions as a function of the bifurcation parameter. In doing this we are projecting the steady-state diagram from three dimensions onto two dimensions. In practice it often suffices to only show one of these figures. For example, we might be more interested in the x component of the steady-state solution.

3.4.2 Bifurcation points for a planar system

We learnt in chapter 1.5 that the steady-state problem

$$f\left(x, \mu\right) = 0,$$

has three kinds of bifurcation points. These are: the limit point bifurcation, the transcritical bifurcation and the pitchfork bifurcation.

The planar steady-state problem

$$\begin{aligned}f\left(x, y, \mu\right) &= 0,\\ g\left(x, y, \mu\right) &= 0,\end{aligned}$$

has the same three kinds of bifurcation points. These bifurcation points will be referred to as *static bifurcation points*. The reasons for this will become apparent in chapter 6.

There is one note of caution which should be recognised when we project a steady-state diagram from three to two dimensions. For a model comprising a single variable the presence of any bifurcation point (limit-point, transcritical, pitchfork) can be determined 'visually' from the steady-state diagram. This is not true for systems containing two, or more, variables: we can only recognise the presence of a limit-point bifurcation 'visually'. The presence of either a transcritical or a pitchfork bifurcation on a steady-state diagram may be an artifact of the projection from three to two dimensions.

3.5 Stability using a Liapunov function

3.5.1 The concept of stability revisited

In section 3.3 we learnt how to determine the stability of a steady-state solution by finding the eigenvalues of the Jacobian matrix; this is known as linearised stability. A steady-state solution was termed stable if 'sufficiently near' initial conditions were attracted to the steady-state solution; this requires that both the eigenvalues of the Jacobian matrix have negative real part. A steady-state solution was termed unstable if at least one of the eigenvalues had positive real part. In the linear system 'unstable' means that the distance between the steady-state solution and the solution of the linear system increases to infinity as time increases towards infinity. This notion of stability/instability is sometimes called *local stability* because it tells us that a steady-state solution is stable to 'sufficiently small' perturbations.

The linearised stability analysis of section 3.3: shows that a steady-state solution is asymptotically stable if the real parts of all the eigenvalues of the Jacobian matrix are negative; the steady-state solution is unstable if the real part of at least one eigenvalue is positive. As noted in remark 3.2 the stability of a steady-state solution is undetermined when either the largest eigenvalue has real-part zero or the eigenvalues are purely imaginary. Linearised stability does *not* tell us how small a perturbation has to be for the system to return to the steady-state solution.

An alternative approach to determine stability is to use a *Liapunov function.* Rather than examining a linearised version of system (3.1), i.e. system 3.3, this seeks to directly determine the stability of the non-linear system (3.1). This method has the advantage that, in principle, it can be used to determine the stability of a steady-state solution when linearised stability fails to do so. Furthermore, it can be used to estimate how large a perturbation can be and for the system to still return to a steady-state solution. The disadvantage of this method is that in order to use we have to construct a Liapunov function. Unfortunately, there is no general technique that can be used to construct a Liapunov function; for any given problem, it is a matter of trial and error.

Before we proceed, we need to refine our definition of *stability.* Roughly speaking[2], a steady-state solution (x^*, y^*) is *neutrally stable* if solutions starting "close" to (x^*, y^*) remain close to the steady-state solution for all later times[3]. A neutrally stable solution is said to be *asymptotically stable* if nearby solutions actually converge to the steady-state solution as $t \to \infty$[4]. An asymptotically stable steady-state solution is sometimes referred to as being an *attractor* of the system. A solution which is not stable is said to be *unstable.*

Question 3.7 (Neutral stability) *Consider the system [54, example 1 on page 132]*

$$\begin{aligned}\dot{x} &= -y^3,\\ \dot{y} &= x^3.\end{aligned}$$

(a) What can you deduce about the stability of the trivial steady-state solution from a linearised stability analysis?

(b) By considering the quotient differential equation $(\frac{dy}{dx})$ *explain why the steady-state solution is neutrally stable, but not asymptotically stable.*

3.5.2 Liapunov stability theorems

In this section we provide some examples of Liapunov stability (and instability) theorems and show how they may be used to deduce stability (instability) results. We also outline some of the disadvantages of this approach.

Theorem 3.1 (From [54], theorem 3 pages 130–131) *Consider the system* (3.1). *Let* $\bar{\mathbf{x}} = (x^*, y^*)$ *be a steady-state solution of system* (3.1) *and let* $V : U \to \mathbb{R}$ *be a* $\mathbf{C}^1$ *function defined on some neighbourhood* U *of* $\bar{\mathbf{x}}$ *such that* $V(\bar{\mathbf{x}}) = 0$ *and* $V(\mathbf{x}) > 0$ *if* $(x, y) \neq (x^*, y^*)$. *Then if:*

(i) $\dot{V}(\mathbf{x}) \leq 0$ *in* $U - \{\bar{\mathbf{x}}\}$ *then* $\bar{\mathbf{x}}$ *is neutrally stable;*

(ii) $\dot{V}(\mathbf{x}) < 0$ *in* $U - \{\bar{\mathbf{x}}\}$ *then* $\bar{\mathbf{x}}$ *is asymptotically stable, that is all solutions of system* (3.1) *with initial conditions in* U *satisfy* $\mathbf{x}(t) \to \bar{\mathbf{x}}$ *as* $t \to \infty$.

(iii) $\dot{V}(\mathbf{x}) > 0$ *in* $U - \{\bar{\mathbf{x}}\}$ *then* $\bar{\mathbf{x}}$ *is unstable*

[2] These terms can be defined in a rigorous manner.
[3] In some books neutral stability is called 'stable in the sense of Liapunov' or 'Liapunov stability'.
[4] From now on by 'stable' steady-state solution we will mean 'asymptotically stable' steady-state solution.

This theorem provides an elegant and simple characterisation of the dynamics of the system (3.1). However, it is reliant upon finding a function (V) with the specified properties. Herein lies the difficulty in applying this theorem — there are no general methods for finding such functions. Considerable ingenuity is usually required.

Remark 3.5 *The function V is referred to as a Liapunov function and*

$$\dot{V}(\mathbf{x}) = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y}$$

is the derivative of the function V along the solution curves of (3.1)*. Consequently, if $\dot{V}(\mathbf{x})$ is negative, then the function V decreases along the solutions of* (3.1)*.*

Example 3.5 *Consider the system [37, based upon example 10.1 on pages 267-268]*

$$\begin{aligned} \dot{x} &= -y - x^3, \\ \dot{y} &= x - y^3. \end{aligned}$$

(i) What does a linear stability analysis tell you about the stability of the origin?

(ii) Investigate the stability of the origin using the function

$$V = ax^2 + by^2,$$

where the values for a and b are to be established.

Solution

(i) We have

$$\begin{aligned} J(x,y) &= \begin{pmatrix} -3x^2 & -1 \\ 1 & -3y^2 \end{pmatrix}, \\ J(0,0) &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

The characteristic polynomial for this matrix is

$$\lambda^2 + 1 = 0.$$

Hence the eigenvalues are

$$\lambda_{\pm} = \pm i.$$

Thus linearised stability provides no information about the stability of the trivial solution.

(ii) First we note that if $a > 0$ and $b > 0$ then

$$V(x,y) > V(0,0)\ \forall\ (x,y) \neq (0,0).$$

Similarly we note that if $a < 0$ and $b < 0$ then

$$V(x,y) < V(0,0)\ \forall\ (x,y) \neq (0,0).$$

We have

$$V(x,y) = ax^2 + by^2.$$

Thus

$$\begin{aligned}\dot{V} &= 2ax\dot{x} + 2by\dot{y},\\ &= 2ax\left(-y - x^3\right) + 2b\left(x - y^3\right),\\ &= -2by^4 + 2\left(b - a\right)xy - 2ax^4.\end{aligned}$$

Taking $a = b = c > 0$ we have

$$\begin{aligned}\dot{V} &= -2c\left(y^4 + x^4\right),\\ &< 0\,\forall\,(x, y) \neq (0, 0).\end{aligned}$$

Thus from theorem 3.1 (ii) we deduce that the trivial solution is asymptotically stable. In fact the Liapunov function we have found for this example provides even more information — see remark 3.8.

□

Next year... generalise this example to the system

$$\begin{aligned}\dot{x} &= d\left(-y - x^3\right),\\ \dot{y} &= d\left(x - y^3\right).\end{aligned}$$

Example 3.6 (from [65], page 13) *Consider the system*

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= -x + \epsilon x^2 y.\end{aligned}$$

It is easy to verify that $(x, y) = (0, 0)$ is the only steady-state of this system and that the eigenvalues, $\lambda = \pm i$, have real part zero.

By considering the Liapunov function $V(x, y) = \left(x^2 + y^2\right)/2$ show that the trivial solution is neutrally stable for $\epsilon < 0$.

Solution Let $V(x, y) = \left(x^2 + y^2\right)/2$. Clearly $V(0, 0) = 0$ and $V(x, y) > 0$ in any neighbourhood of $(0, 0)$. Then

$$\begin{aligned}\dot{V}(x, y) &= x \times \dot{x} + y \times \dot{y},\\ &= xy + y\left(-x + \epsilon x^2 y\right),\\ &= \epsilon x^2 y^2.\end{aligned}$$

Then, by theorem 3.1 (a) $(0, 0)$ is neutrally stable for $\epsilon < 0$. With a little work it is possible to show that $(0, 0)$ is globally asymptotically stable for $\epsilon < 0$. □

Next year. Put this as an example for LaSalle's theorem.

Question 3.8 *Can you construct a function V to show that the steady-state solution of example 3.6 is globally asymptotically stable for $\epsilon < 0$?*

Hint. Return to this question after you have finished the chapter — you might gain a few ideas along the way!

The signs of the inequalities in theorem 3.1 can be reversed. For example.

Theorem 3.2 ([25], theorem A2.2.6 on page 150) *Consider the system* (3.1)*. Let $\bar{\mathbf{x}} = (x^*, y^*)$ be a steady-state solution of system* (3.1) *and let $V : U \to \mathbb{R}$ be a $\mathbf{C}^1$ function defined on some neighbourhood U of $\bar{\mathbf{x}}$ such that $V(\bar{\mathbf{x}}) = 0$ and $V(\mathbf{x}) < 0$ if $\mathbf{x} \neq \bar{\mathbf{x}}$. Then if $\dot{V}(\mathbf{x}) < 0$ in $U - \{\bar{\mathbf{x}}\}$ then $\bar{\mathbf{x}}$ is unstable.*

Parts (i) & (ii) of theorem 3.1 can be similarly be reversed in the obvious manner. See, for example, [57, page 239].

Remark 3.6 *The method of determining the stability of a steady-state solution through the use of a Liapunov function is sometimes called 'Liapunov's direct method' or 'Liapunov's second method'.*

The construction of an appropriate Liapunov function allows the stability of a steady-state solution to be determined without having to determine the eigenvalues of the Jacobian. Furthermore, it an appropriate Liapunov function determines stability when linearised stability does not. However, the use of technique is restricted by the difficulty of finding a function V satisfying the required properties of theorem 3.1 or theorem 3.2 for system (3.1). The simplest procedure is often trial and error, although for mechanical and structural systems the total energy of the system is often a good candidate for a Liapunov function.

Note that the neighbourhood U partly answers the question on how small 'small' is. If the steady-state solution $(\bar{\mathbf{x}})$ is asymptotically stable then the solution of the differential equation starting at any point in the region $U - \{\bar{\mathbf{x}}\}$ approaches $\bar{\mathbf{x}}$ as $t \to \infty$. The region U does not necessarily contain every initial condition whose solution converges to the steady-state solution.

Definition 3.4 (Basin of attraction) *Suppose that the steady-state solution $\bar{\mathbf{x}}$ is asymptotically stable. Then, by definition, there is a neighbourhood, N of $\bar{\mathbf{x}}$, such that any solution curve starting in N tends towards $\bar{\mathbf{x}}$ as $t \to \infty$. The union of all solution curves that tend towards $\bar{\mathbf{x}}$ (as $t \to \infty$) is called the basin of attraction, or just basin, of $\bar{\mathbf{x}}$. The basin of attraction of $\bar{\mathbf{x}}$ is denoted $B(\bar{\mathbf{x}})$.*

In some mathematical models it is important to determine the basin of attraction of a stable steady-state solution. For example, the steady-state solution may represent a desired equilibrium state of a physical system. The extent of the basin tells us how large a perturbation from equilibrium can be allowed and still be sure that the system will return to equilibrium. A region within the basin of attraction can be found by finding an appropriate Liapunov function. However, the region U established by the Liapunov function may be a very poor approximation to the basin of attraction.

Although there are some practical systems where information gained from the use of Liapunov functions is important, in many (most?) practical systems this is not the case: stability can be determined via linear stability analysis and basins of attraction are unimportant.

The practical importance of using a Liapunov function to establish the basin of attraction is often dramatically over-stated

Remark 3.7 *If we can choose $U = \mathbb{R}^2$ in case (iii) of theorem 3.1, then the basin of attraction of $\bar{x}$ is $\mathbb{R}^2$ and the steady-state solution is said to be globally asymptotically stable: all solutions remain bounded and in fact approach the steady-state solution $\bar{x}$ as $t \to \infty$.*

Remark 3.8 *In example ii we showed that the trivial solution of the system*

$$\dot{x} = -y - x^3,$$
$$\dot{y} = x - y^3.$$

is asymptotically stable. In fact our calculation shows that globally asymptotically stable

In order to show some of the 'fiddling' that is required to construct a Liapunov function we consider an extension of the system considered in example 3.1

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y,$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{k}{m}\left(x + x^3\right) - \alpha y, \qquad \alpha > 0$$

In this system the term $-\alpha y$ represents damping.

Question 3.9 (from [33], page 5)

1. *Consider the function*
$$V_1(x,y) = \frac{my^2}{2} + k\left(\frac{x^2}{2} + \frac{x^4}{4}\right).$$
   (a) *Show that*
$$\dot{V}_1 = -\alpha m y^2.$$
   (b) *What can you deduce about the stability of the trivial solution from the function* V_1*?*

2. *Consider the modified function*
$$V_2(x,y) = \frac{my^2}{2} + k\left(\frac{x^2}{2} + \frac{x^4}{4}\right) + \beta\left(xy + \frac{\alpha x^2}{2}\right),$$
   where the value of β *is to be determined. Show that*

   (a) *The trivial steady-state solution is a minimum of the function* V_2*. (See appendix E).*

   (b)
$$\dot{V}_2 = -\beta\frac{k}{m}\left(x^2 + x^4\right) - (\alpha m - \beta)\, y^2 < 0,$$
   for β *sufficiently small.*

The function V_1 shows that the steady-state solution $(x,y) = (0,0)$ is *neutrally stable.* However, as $\dot{V} = 0$ along the x-axis we can not deduce that the steady-state solution is *asymptotically stable.* The function V_2 shows that the steady-state solution is in fact asymptotically stable.

Question 3.9 shows both the power and the weakness of the method. We can establish a *global result* about the stability of a steady-state solution. However, producing the form of the function V that decides the stability in this way is the difficult practical feature on the method: could you have successfully modified our first function (V_1) to establish this result?

Remark 3.9 *In problems where the functions* $f(x,y)$ *and* $g(x,y)$ *in system 3.1 are polynomial functions a good guess is to try*
$$V(x,y) = ax^2 + bxy + cy^2,$$
where the value of the coefficients a, b and c are determined by inspection. [32, page 108]

Note that the requirement that the trivial solution is a local minimum, or a local maximum, of the function V given in remark 3.9 imposes restrictions on the values that the coefficients a, b and c can take — see question E.1 in appendix E.

Would be good to have a nice example here

Remark 3.10 *For problems with multiple equilibria, such as example 3.2, local Liapunov functions can be sought to establish the stability of a steady-state solution and to investigate how small 'small' is.*

Different versions of theorem 3.1 (iii) are available to establish the instability of a steady-state solution. One such is the following.

Theorem 3.3 ([37], theorem 10.3 on page 272.) *Consider the system* (3.1)*. Let* $\bar{\mathbf{x}} = (x^*, y^*)$ *be a steady-state solution of system* (3.1) *and let* $V : U \to \mathbb{R}$ *be a* $\mathbf{C}^1$ *function defined on some neighbourhood* U *of* $\bar{\mathbf{x}}$ *such that*

(i) $V(\bar{\mathbf{x}}) = 0$ *and in every neighbourhood of* $\bar{\mathbf{x}}$ *there exists at least one point* $\mathbf{x}$ *at which* $V(\mathbf{x}) > 0$*.*

(ii) $\dot{V}(\mathbf{x}) > 0$ *in* $U - \{\bar{\mathbf{x}}\}$.

Then $\bar{\mathbf{x}}$ *is unstable.*

Next year. How do you know that you are trying to prove that, the solution is unstable?

Remark 3.11 *Typically, the function* V *in theorem 3.3 will take both positive and negative values close to the steady-state solution* $\bar{x}$. *Functions of the type* xy *or* $x^2 - y^2$ *have this property and may be successful.*

Example[[37]], example 10.4 on page 273] Show that the origin is an unstable solution of the equation

$$\ddot{x} + \sin(x)\,\dot{x} - x = 0.$$

Solution The equivalent system is

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= x - y\sin x.\end{aligned}$$

Consider the Liapunov function

$$V = xy.$$

Then

$$\dot{V}(x,y) = x^2 + y^2 - xy\sin x.$$

We show that this function has a minimum at the origin. Define

$$\begin{aligned}A &= \frac{\partial^2 \dot{V}}{\partial x^2}(0,0) = 2,\\ B &= \frac{\partial^2 \dot{V}}{\partial x \partial y}(0,0) = 0,\\ C &= \frac{\partial^2 \dot{V}}{\partial y^2} \qquad = 2.\end{aligned}$$

Then

$$AC - B^2 = 4 > 0 \quad and A > 0$$

so that the origin is a local minimum of the function $\dot{V}$. This implies that $\dot{V} > 0$ in a neighbourhood of the origin (excluding the origin). Consequently the properties of theorem 3.3 are satisfied and the trivial solution is therefore unstable. □

Example 3.7 *Consider the system [32, page 110, problem 11]*

$$\begin{aligned}\dot{x} &= x - y + x^2\sin(y),\\ \dot{y} &= -2y + x^3;\end{aligned}$$

1. *Using linearised stability analysis show that the trivial solution is unstable.*
2. *What can you deduce about the stability of the origin using the Liapunov function*

$$V = x^2 - y^2$$

Solution

1. The Jacobian matrix for the system is given by

$$\begin{aligned} J(x,y) &= \begin{pmatrix} 1+2x\sin(y) & = -1 + x^2\cos(y) \\ 3x^2 & -2 \end{pmatrix}, \\ J(0,0) &= \begin{pmatrix} 1 & -1 \\ 0 & -2 \end{pmatrix} \\ \operatorname{tr} J(0,0) &= -1, \\ \det J(0,0) &= -2. \end{aligned}$$

   As $\det J < 0$ the trivial solution is unstable.

2. A straight forward calculation shows that

$$\dot{V} = -2yx^3 + 2x^3\sin(y) + 2x^2 - 2xy + 4y^2$$

   It seems difficult to find a specific neighbourhood of the origin where $\dot{V} > 0$. Let us therefore assume that $|x| \ll 1$ and that $y\,|y| \ll 1$. Thus

$$\begin{aligned} \dot{V} &\approx 2x^2 - 2xy + 4y^2, \\ &= 2\left[x^2 - xy + 2y^2\right], \\ &= \left(x - \frac{y}{2}\right)^2 + \frac{7}{4}y^2 && > 0 \quad \text{when } (x,y) \neq (0,0). \end{aligned}$$

   Thus the origin is unstable.

□

Question 3.10 *Is it possible to use theorem 3.1 (iii) to show that the origin is unstable in the previous question using a Liapunov function of the form*

$$V(x,y) = ax^2 + bxy + cy^2$$

3.5.3 LaSalle's Theorem

The following question provides motivation for an extension of the basic stability theorem 3.1.

Question 3.11 (Neutral stability for a mechanical system) *Consider the equation*

$$\ddot{x} + f(x)\dot{x} + g(x) = 0$$

with the equivalent system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -g(x) - f(x)\,y \end{aligned}$$

Suppose that the functions $f(x)$ and $g(x)$ satisfy the following requirements:

- $f(x) > 0$ *for* $x \neq 0$ *(the damping is positive).*
- $g(0) = 0$.
- $xg(x) > 0$ *for* $x \neq 0$.

Consider the function [57, example page 242]

$$V(x,y) = \frac{y^2}{2} + G(x),$$

where

$$G(x) = \int_0^x g(u)\, du$$

1. *Show that* $V(0,0) = 0$ *and that* $V(x,y) > 0$ *if* $(x,y) \neq (0,0)$.
2. *Show that*
$$\dot{V}(x,y) = -f(x)\, y^2 \leq 0.$$
3. *What can you conclude about the stability of the origin?*

Note that the problem considered in example 3.9 is a special case of this system with $g(x) = \frac{k}{x}\left(x + x^3\right)$ and $f(x) = \alpha$.

Theorem 3.4 (LaSalle) *Assume* $V(x) \geq 0$ *and* $\dot{V} \leq 0$ *for all* x. *Let* E *be the locus* $\dot{V} = 0$, *and let* M *be the union of all trajectories that remain in* E *for all* t. *Then all solutions of* $\dot{x} = X(x)$ *that are bounded for* $t > t_1$, *for some* t_1, *approach* M *as* $t \to \infty$. *(Lagrange stability).*

Example 3.8 (Application of LaSalle's Theorem to a mechanical system, [[57], pages 252 & 253) *] We reconsider the mechanical model from question 3.11 with the additional assumption that*

$$|F(x)| = \left| \int_0^x f(u)\, du \right| \to \infty \qquad as \qquad |x| \to \infty.$$

1. *Show that the region* $R(A, \alpha)$ *defined by*
$$V(x,y) \leq A,$$
$$[y + F(x)]^2 \leq a^2,$$
is positively invariant.
2. *Hence deduce that the origin is globally asymptotically stable.*

1. Because $\dot{V} \leq 0$, a solution starting in R cannot cross $V = A$. By choosing a sufficiently large, we can ensure that for a part of the boundary of R, that is, for $y + F(x) = \pm a$, we have
$$\frac{\mathrm{d}}{\mathrm{d}t}[y + F(x)]^2 = -2\,|g(x)| < 0.$$
Thus for any A and for sufficiently large a, every solution in R remains in R.
2. From the definition of V it follows that $\dot{V} = 0$ only on $y = 0$ and, possible, $x = 0$. Therefore, if the origin is excluded, no solution remains on the x and $y$axes. Thus M is the origin, and by the foregoing theorem all solutions bounded for $t > t_1$, for some t_1, approach M as $t \to \infty$.

   From the first part of this question we know that the region R is positively invariant. Thus all solutions starting in R are bounded for $t > 0$ and approach the origin as $t \to \infty$. Furthermore, the region R can be extended to the entire plane, thus all solutions are bounded and we have global asymptotic stability.

3.5.4 Questions on stability using a Liapunov function

1. Consider the system from example 3.1:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y,$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{k}{m}\left(x + x^3\right).$$

   (a) Find the steady-state solution of this system and determine its linear stability.

   (b) The associated total energy of this system is given by

$$V(x, y) = \frac{my^2}{2} + k\left(\frac{x^2}{2} + \frac{x^4}{4}\right).$$

   Using this function what can you deduce about the stability of the steady-state solution? [33, page 5].

2. (a) Consider the system [32, page 105]

$$\dot{z}_1 = -2z_1z_2^2 - z_1^3,$$
$$\dot{z}_2 = -z_2 + z_1^2 z_2.$$

   (i) Show that the origin is the only steady-state of this system and determined its eigenvalues.

   (ii) Using the function

$$V = az_1^2 + z_2^2,$$

   where the value for a is to be established, what can you deduce about the stability of this system?

   (b) Consider the system [37, based upon example 10.2 on page 271]

$$\dot{x} = -x - 2y^2,$$
$$\dot{y} = xy - y^3.$$

   Investigate the stability of the origin using the function

$$V = ax^2 + by^2,$$

   where the values for a and b are to be established.

   (c) In question 3.4 (section 3.3.3) we showed that the system

$$\dot{x} = x + k\left(x^2 + y^2\right)x,$$
$$\dot{y} = y + k\left(x^2 + y^2\right)y,$$

   had a locally stable (unstable) steady-state solution $(x, y)\ (0, 0)$ depending upon whether $k < 0$ ($k < 0$). What information can you obtain about this system using the Liapunov function $V = ax^2 + by^2$, where the values for $a > 0$ and $b > 0$ are to be determined? [37, Example 10.7 on page 280]

3. Consider the system

$$\dot{z}_1 = z_2,$$
$$\dot{z}_2 = -\omega^2 z_1 - \nu z_1^2 z_2.$$

   (a) By considering the function

$$V = \frac{1}{2}\left(z_2^2 + \omega^2 z_1^2\right)$$

   show that the steady-state solution is (neutrally) stable.

   (b) Can you construct a suitable Liapunov function to show that that the steady-state solution is either asymptotically stable or globally asymptotically stable? (c.f. question 3.9).

[32, page 105]

4. Consider the system

$$\dot{x} = y,$$
$$\dot{y} = -\left[f(x, y)\, y + \lambda x\right],$$

where the function $f(x, y) \geq 0$ in a neighbourhood of the origin and $\lambda > 0$. This system is equivalent to the second-order differential equation

$$\ddot{x} + f(x, \dot{x})\,\dot{x} + \lambda x = 0,$$

representing a linear mass-spring system modified by a nonlinear damping term, giving positive damping for small amplitudes.

(a) By considering the function

$$V = ax^2 + by^2,$$

where the values $a > 0$ and $b > 0$ are to be established, show that the origin is neutrally stable.

(b) Can you construct a suitable Liapunov function to show that that the steady-state solution is either asymptotically stable or globally asymptotically stable? (c.f. question 3.9).

[37, slightly adapted from example 10.3, page 271]

5. Consider the system

$$\dot{x} = y,$$
$$\dot{y} = -x + x^3 - 2\mu y,$$

which is known as the Duffing oscillator.

(a) Find the steady-state solutions of the Duffing oscillator.

(b) What can you deduce about the stability of the trivial solution using the Liapunov function

$$V = \frac{1}{2}x^2 + \frac{1}{2}y^2 - \frac{1}{4}x^4$$

(c) Can you construct a suitable Liapunov function to show that that the trivial steady-state solution is asymptotically stable? (c.f. question 3.9).

[47, example 1.14]

6. For each of the following systems:

$$(a) \quad \dot{z_1} = -2z_1 + z_2^3, \qquad \dot{z_2} = z_1 - z_2 + z_1 z_2^2;$$
$$(b) \quad \dot{z_1} = 3z_1 + 2z_2 + z_2^2, \dot{z_2} = -10z_1 - 5z_2 - z_1^2 z_2,$$

use a suitable Liapunov function $V(z_1, z_2)$ to determine the stability of the zero solution. In each case start try

$$V = az_1^2 + 2bz_1z_2 + cz_2^2$$

and determine the coefficients a, b and c to give V and $\dot{V}$ suitable properties. Compare with the results obtained using the principle of linearised stability.

Hint. Choose a, b and c so that $\dot{V}$ is approximately $\pm\left(z_1^2 + z_2^2\right)$, with the negative sign for a stable steady-state solution and a positive sign for an unstable steady-state solution. [32, page 110, problem 11].

7. The origin is a center for the linear system

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x},$$

where $\mathbf{x}$ is the vector $\begin{pmatrix} x \\ y \end{pmatrix}$. The addition of nonlinear terms to the right-hand side of this linear system changes the stability of the origin. Use an appropriate Liapunov function to establish the following facts.

(a) The origin is asymptotically stable for the system

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} -x^3 - xy^2 \\ -y^3 - yx^2 \end{bmatrix}.$$

(b) The origin is unstable for the system

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} x^3 + xy^2 \\ y^3 + yx^2 \end{bmatrix}.$$

(c) The origin is neutrally stable for the system

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} -xy \\ x^2 \end{bmatrix}.$$

What are the solution curves in this case?

[54, problem 4 on page 135]

8. Use appropriate Liapunov functions to determine the stability of the steady state solutions of the following systems

$$\begin{aligned}
&(a) \quad \dot{x_1} = -x_1 + x_2 + x_1x_2, && \dot{x_2} = x_1 - x_2 - x_1^2 - x_2^3,\\
&(b) \quad \dot{x_1} = x_1 - 3x_2 + x_1^3, && \dot{x_2} = -x_1 + x_2 - x_2^2,\\
&(c) \quad \dot{x_1} = -x_1 - 2x_2 + x_1x_2^2, && \dot{x_2} = 3x_1 - 3x_2 + x_2^3,\\
&(d) \quad \dot{x_1} = -4x_2 + x_1^2, && \dot{x_2} = 4x + 1 + x_2^2.
\end{aligned}$$

[54, problem 5 on page 135]

9. Consider the system

$$\begin{aligned}\dot{x} &= x^2 - xy,\\ \dot{y} &= -y + x^2.\end{aligned}$$

(a) Find the steady-state solutions of this system and determine their linearised stability.

(b) By using a coordinate transformation $X = x - 1$, $Y = y - 1$ show that the system can be written in the form

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} X^2 - XY \\ X^2 \end{pmatrix}$$

By finding a suitable Liapunov function show that the zero solution of the transformed system is stable.

(Based upon [33, equation (1.8.20) on page 54]

10. (a) By finding an appropriate Liapunov function show that the trivial steady-state solution of the system

$$\begin{aligned}x' &= -2x - y^2,\\ y' &= -y - x^2\end{aligned}$$

is asymptotically stable.

(b) Using your Liapunov function find $\delta > 0$ as large as you can such that open disk of radius δ and center $(0, 0)$, that is the set of points satisfying $x^2 + y^2 < \delta^2$, is contained within the basin of attraction of the trivial steady-state solution.

[34, problem 1 on page 198]

11. (a) By finding an appropriate Liapunov function show that the trivial solution of the system

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= -\omega^2 x - 2\mu y,\end{aligned}$$

is asymptotically stable for $\mu > 0$. Is it globally asymptotically stable? [47, Problem 1.5 on page32]

(b) By finding an appropriate Liapunov function show that the trivial solution of the system

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= -x - \mu y^3,\end{aligned}$$

is asymptotically stable and determine the domain of attraction. [47, Problem 1.6 on page33]

(c) By finding an appropriate Liapunov function show that the trivial solution of the system

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= -x - x^3 - \mu y \left|x\right|,\end{aligned}$$

is asymptotically stable and determine the domain of attraction.
Hint. Start with the function $V = \frac{1}{2}y^2 + \frac{1}{2}x^2 + \frac{1}{4}x^4$. [47, Problem 1.6 on page33]

12. Consider the system

$$\begin{aligned}\dot{x} &= -x\left(1 - x^2 - y^2\right),\\ \dot{t} &= -y\left(1 - x^2 - y^2\right).\end{aligned}$$

(a) By using the function

$$V = ax^2 + by^2,$$

where the values $a > 0$ and $b > 0$ are to be established. show that the origin is asymptotically stable.

(b) Deduce that all points in the disc $x^2 + y^2 < 1$ are in the basin of attraction of the origin.

(c) Show that there are no points outside the disk $x^2 + y^2 < 1$ that are in the basin of attraction of the origin.
Hint. Transform to polar co-ordinates.

13. One form of the Van der Pol equation is

$$\begin{aligned}\dot{y} &= x,\\ \dot{x} &= \epsilon\left(1 - y^2\right)x - y.\end{aligned}$$

By applying LaSalle's Theorem, theorem 3.4, and using the Lyapunov function

$$V\left(x, y\right) = \frac{x^2 + y^2}{2},$$

show that the origin is asymptotically stable for all solution starting in the region $x^2 + y^2 < 3$.

3.6 Conclusions

3.6.1 Summary

Given a system of two first-order differential equations you should start your analysis by finding finding the steady-state solutions of (3.1) and determining their stability. In almost all practical applications it suffices to find local stability using the method of linearised stability. Both the method of linearised stability and Liapunov's direct method generalise from a system of two first-order differential equations to a system of n first-order differential equations.

A major difference between a single autonomous differential equation and a system of two, or more, autonomous differential equations is that periodic solutions are impossible in the former, but may occur in the latter. A *periodic solution* of the planar system (3.2) is a solution $\mathbf{x}$ that is not a steady-state of the system and which has the property that

$$\mathbf{x}\left(t + T\right) = \phi\left(t\right),$$

for some $T > 0$.

Question 3.12 *Show that the model for a linear oscillator*

$$\ddot{x} + \omega^2 x = 0$$

has periodic solutions.

Unfortunately, in general it is impossible to tell by looking at the right-hand side of system (3.1), whether this system has periodic solutions. In chapter 5 we describe two methods that can be used to prove the non-appearance of periodic solutions. In chapter 6 we describe a method that can be used to prove the appearance of periodic solutions under small perturbation of the system (for example, by variation of parameters on which the system depends).

3.6.2 Historical comments

Aleksandr Mikhailovich Liapunov[5] introduced the method of determining stability using a Liapunov function (Liapunov's second method) in his 1892 doctoral thesis: *The general problem of the stability of motion.* Liapunov's first method uses perturbation and other procedures to study the solution and stability of a system.

In his thesis Lyapunov proved the following linear stability theorem

Theorem 3.5 (Lyapunov, 1892) *Consider a dynamical system defined by*

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n,$$

where f is two times continuously differentiable. Suppose that it has a steady-state solution $\bar{x}$ (i.e. $f(\bar{x}) = 0$), and denote by J the Jacobian matrix of $f(x)$ evaluated at the steady-state solution, $J = f_x(\bar{x})$. Then

1. *$\bar{x}$ is asymptotically stable if all eigenvalues of J satisfy $Re\,\lambda < 0$.*
2. *$\bar{x}$ is unstable if there is one, or more, eigenvalues of J satisfying $Re\,\lambda > 0$.*

This theorem generalises the method described in section 3.3 from a system of two differential equations to a system of n differential equations. It can easily be proved for a linear system

$$\dot{x} = Jx, \quad x \in \mathbb{R}^n,$$

by its explicit solution in a basis where A has Jordan normal form. For a general nonlinear system it is proved by constructing an appropriate Liapunov function near the steady-state solution. More precisely, by a shift of coordinates, the steady-state solution can be placed at the origin, $\bar{x} = 0$. Asymptotic stability in a neighbourhood of the origin can then be established by using a quadratic form as a Liapunov function. Actually, the Liapunov function is the same for both linear and nonlinear systems and is fully determined by the Jacobian matrix J. [38, page 21]. [6]

3.6.3 Definite and semidefinite functions

Theorems using Liapunov functions are often stated in terms of functions being definite or semidefinite.

Definition 3.5 (Definite function) *A function is definite (positive or negative) in a region if it has the same sign throughout the region and vanishes only for zero values of the variables.*

Example 3.9 (Negative definite function) *The function*

$$V(x, y) = -\left(x^2 + y^2\right)$$

is negative-definite.

[5] Russian mathematician (1857–1918). The surname is often spelt Lyapunov. The biographical account is primarily based upon the `MacTutor` biography.

[6] Mathematical details come from [38, page 21].

Definition 3.6 (Semidefinite function) *A function is semidefinite (positive or negative) in a region if it has the same sign throughout the region and vanishes for zero values of the variables and at other points.*

Example 3.10 (Positive semi-definite function) *The function*

$$V(x,y) = x^4$$

is positive semi-definite, since it vanishes along the y-axis; not just at the origin.

Definition 3.7 (Indefinite function) *A function is indefinite (or variable) if it is neither definite nor semidefinite.*

Question 3.13 *Rewrite theorems 3.1, 3.2, 3.3a & 3.4 so that the properties of the Lyapunov function (V) and its derivative ($\dot{V}$) are replaced by those of definite/semi-definite functions.*

3.6.4 Further reading

The theorems that we have stated for Liapunov functions, theorems 3.1 & 3.3, can be extended. For example, they can be extended to non-autonomous systems. See [57, pages 250–252]. For examples of other extensions see [34, Theorem 2 on pages 196–198] and the following references.

Barnett, S., and Storey, C. (1970). *Matrix methods in stability theory*, chapter 5. London: Nelson.

Burton, T.A. (1969). On the construction of Lyapunov functions. *SIAM J. Appl. Math*, **17**, 1078–1085.

Hagedorn, P. (1988). *Non-Linear Oscillations*. Clarendon Press, Oxford, England. (A detailed exposition)

Krasovskii, N.N. (1963). *Stability of Motion*. Stanford University Press, Stanford, California. (Covers the existence of Liapunov functions and applications of Liapunov's second method in length).

LaSalle, J.P. and Lefschetz, S. (1961). *Stability by Liapunov's Direct Method with Applications*. Academic Press: New York.

Rosen, R. (1970). *Dynamical system theory in biology* Vol 1., chapter 3. New York: John Wiley and Sons.

Rouche, N., Habets, P. and Laloy, M. (1977). *Stability Theory by Lyapunov's Direct Method*. Springer: New York.

3.7 Maple commands

3.8 Revision of key ideas

The following questions are about the key ideas in this chapter.

3.9 Questions on steady-states and stability

1. The following model for the concentration of a nutrient (S) and microorganism feeding upon the nutrient (X) in a continuously stirred tank reactor has been proposed [62]

$$\begin{aligned}\frac{\mathrm{d}S}{\mathrm{d}t} &= \frac{1}{\tau}\left(S_0 - S\right) - \frac{\mu_{\max}}{\alpha}\cdot XS,\\ \frac{\mathrm{d}X}{\mathrm{d}t} &= \mu_{\max}XS - \frac{1}{\tau}X.\end{aligned} \tag{3.10}$$

In this model: S_0 is the concentration of nutrient in the incoming medium; t is time; α is known as the yield constant; μ_{max} is the growth rate of microorganisms feeding upon nutrients; τ is the residence time, which is the main parameter that is experimentally controllable. All constants in this model are non-negative.

(a) Show that if the initial values of the state-variables satisfy $S(t=0) \geq 0$ and $X(t=0) \geq 0$ then the quantities $S(t)$ and $X(t)$ are never negative, i.e. show that the positive quadrant is positively invariant.

(b) Find the steady-state solutions of system (3.10). Show that they correspond to a 'washout branch', in which $X^* = 0$, and a 'no-washout branch', in which $X^* \neq 0$.

(c) Determine the stability of the washout branch as a function of the residence time (τ).

(d) (a) Show that along the no-washout branch the substrate concentration is always positive.

(b) Find the values of the residence time over which the concentration of microorganisms is negative.

(c) The no-washout branch is said to be 'physically meaningful' when the values of both components (substrate, microorganisms) are both positive. For what values of the residence time is the no-washout branch physically meaningful?

(e) Show that when the no-washout branch is physically meaningful that it is stable.

(f) Plot the steady-state diagrams (including stability) for the substrate concentration and microorganism concentration as a function of the residence time. For the no-washout branch only plot physically meaningful solutions.

2. The following model has been proposed for continuous fermentation in conditions where the products of fermentation are toxic and limit the fermentation process [40]

$$\begin{aligned} \frac{\mathrm{d}X}{\mathrm{d}t} &= (k_m - qP)\,X - \frac{X}{\tau}, \\ \frac{\mathrm{d}P}{\mathrm{d}t} &= [\alpha\,(k_m - qP) + \beta]\,X - \frac{P}{\tau}. \end{aligned} \tag{3.11}$$

In this model : P is the product concentration ($\mathrm{mg\,ml^{-1}}$); X is the concentration of bacteria in the vessel ($\mathrm{mg\,ml^{-1}}$); k_m is the maximum specific growth rate ($\mathrm{h^{-1}}$); q is the product inhibition constant ($\mathrm{h^{-1}mg^{-1}ml}$); t is time (h), α is a constant associated with growth-associated product formation (—); β is a constant associated with nongrowth-associated product formation ($\mathrm{h^{-1}}$); τ is the residence time, the main parameter that is experimentally controllable, (h). All constants are strictly non-negative.

(a) Find the steady-state solution of the system (3.11) and determine its stability as a function of the residence time.

(b) The model only makes sense if the steady-state solution is non-negative. Does this impose any restrictions on the parameters?

3. The following model for the concentration of a substrate (S) and microorganism (X) feeding upon the substrate in a continuously stirred tank reactor applies when there is a growth limitation due to substrate inhibition [2].

$$\begin{aligned} \dot{X} &= -\frac{1}{\tau}X + \mu X, \\ \dot{S} &= \frac{1}{\tau}(S_0 - S) - \frac{\mu}{\alpha}X, \\ \dot{\mu} &= \frac{\mu_m K_i S}{S^2 + K_i S + K_i K_s}. \end{aligned}$$

In this model S_0 is the concentration of substrate flowing into the reactor, α is the yield constant, μ is the specific growth rate and τ is the residence time, which is the main experimentally controllable parameter. The parameters K_s, K_s and μ_m are kinetic constants.

(a) Show that the steady-state concentration is given by $S_1 = S_0$ and the solutions of the following quadratic equation

$$S^2 + K_i\,(1 - \mu_m \tau)\,S + K_i K_S = 0.$$

(b) For what value(s) of the residence time does a limit-point bifurcation occur in this model?

(c) Draw a steady-state diagram showing how the steady-state substrate concentration varies as a function of the residence time. (Do not show stability). Use the following parameter values [55, figure 17.12]: $\mu_m = 1.0$, $K_s = 1.0$, $K_i = 10$, $S_0 = 28$, $Y = 0.5$.

(d) Determine the stability of the solution branches in your steady-state diagram. (This is more difficult. You will need to find the corresponding steady-state values for X.)

3.10 Things to do

Stability 1. Compare approach to that used in chapter 1 and make sure that the important themes are re-enforced.

2. Introduce the $\bar{x}$ notation for a steady-state solution.
3. Holling-Tanner model.

Liapunov functions

1. Discussion of when does the instability or asymptotic stability of the linearised system imply the same property for the nonlinear system? For use of Liapunov functions to answer this question see [37, chapter 10.4]. Chapter 10.5 from [37] and all the exercises from [37, chapter 10].
2. Include one of the examples from last year's tutorial sheet.
3. Need to look at check question 3 (b), I wasn't happy with the solution.
4. When is the quadratic function $ax^2 + 2bxy + cy^2$ positive definite?
5. More complicated examples of Liapunov functions. [54, page 134].
6. Make the point that as a first step it is useful to consider $b = 0$, but that we can't assume $a = 0$ or $c = 0$.
7. next book to read on Liapunov functions is Wiggins.
8. One zero root. Discussed in Saaty and Bram, pages 246–249.
9. The application of La Salle's Theorem to the mechanical model comes from.
   LaSalle, J.P. Asymptotic Stability Criteria, in "Proceedings of Symposia in Applied Mathematics," volume 13, American Mathematical Society, 1962.
10. A better discussion of La Salle's Theorem
11. Should the extension of example 3.6 be discussed in the section on La Salle's Theorem?

Bifurcations Perhaps in a new chapter. No! At least mention in this chapter that one way for stability to change is for a single eigenvalue to pass through zero. Example of a system where the projected steady-state diagram indicates a branch point where there is not really a branch point. That would be a good idea.

Stability Link to Hyperbolic steady-states. How can stability change?

Maple code Need to write this section.

Key Ideas Need to write this section.

Nomenclature section Perhaps at the end of each chapter, summarising all the new words.

Appendix Finding the minimum of a function of two variables. A revision of multi-variable calculus. e.g. $x^2 + y^2 - xy \sin x$ has a minimum at the origin

Ecology Integrate the content of the file `ecology.tex` into the text at an appropriate point.

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