Appendix B

Taylor series

B.1 Introduction

Values of polynomials, such as $p(x) = x^3 + 2x^2 - 3x + 1$, may be readily calculated for any value of x whereas many other functions, such as $f(x) = \sin x$, cannot be evaluated, for most values of x, without the aid of a calculator. In appendix B.2 we show how to approximate a function of one variable $f(x)$ near a given point a by a polynomial.

Question B.1 *The function* $\sin x$ *can be evaluated using a calculator. But how does the calculator know what the value is?*

B.2 Taylor series expansion of a function of one variable

The Taylor[1] series of degree n that approximates a function of one variable $f(x)$ about the point $x = a$, is given by

$$\begin{aligned} f(x) &= f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \ldots + \frac{(x-a)^n}{n!} f^{(n)}(a) \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x-a)^k , \end{aligned}$$

where $f^{(0)}(x) = f(x)$. A Taylor series is also known as a Taylor polynomial. The special case when $a = 0$ is also called a Maclaurin[2] series.

This equation provides an extremely accurate approximation, for a large class of functions, for values of x near the point a. The accuracy of the approximation decreases as the value for x moves away from the point a, i.e. as the the size of the number $|x - a|$ increases.

It should be noted that when finding the nth Taylor polynomial, the 'n' refers to the *degree of the highest term*, and not to the *number of terms*.

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[1] Brook Taylor (1685–1731) was an English mathematician. His *Methodus Incrementorum Directa et Inversa* (1715) contained the formula now known as Taylor's theorem. The importance of this theorem was not recognised until 1772, when J.L. Lagrange realized its powers. source: wikipedia entry on Taylor.

[2] Colin Maclaurin (1698–1746) was a Scottish mathematician series who published his work on 'Maclaurin series' in *Methodus incrementorum directa et inversa*. The term Maclaurin series was not attributed to Maclaurin because he discovered them, rather they are attributed to him because of his use of them. In particular, he used these series to characterise maxima, minima, and points of inflection for infinitely differentiable functions. As a point of fact, Maclaurin attributes Taylor in his work. In fact earlier mathematicians were aware of the method, although this was not known to Maclaurin. source: wikipedia entry on Maclaurin.

Question B.2 *Let x^* be a number and ξ a variable. Show that the taylor series expansion of the function $f(x^* + \xi)$ near the point x^* is given by*

$$f(x^* + \xi) \approx f(x^*) + f'(x^*)\,\xi + O(\xi^2).$$

B.3 Taylor series expansion of a function of two variables

Taylor's theorem can be extended to the two variable case. Close to $(x,y)=(x_1,y_1)$ we can approximate the surface $z = f(x,y)$ with the tangent plane to the surface at $(x,y)=(x_1,y_1)$.

We extend the definition of Taylor's Theorem to functions of two variables in the following way. let R be some region containing the point $P(a,b)$. Let the point $S(a+h, b+k)$ be in R such that the line segment PS is in R. Let f have continuous first- and second-order partial derivatives in R. Describe the line segment PS parametrically as $x = a+th$, $y = b+tk$, with $0 \le t \le 1$. Now we study the values of $f(x,y)$ on the line segment PS by considering the function.

$$F(t) = f(a+ht, b+kt).$$

We have

$$F'(t) = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}$$

and F' is continuous and differentiable on $[0,1]$ because f_x and f_y are (by definition above).

Also

$$\begin{aligned} F'' &= h\frac{\partial F'}{\partial x} + k\frac{\partial F'}{\partial y} \\ &= h\left(h\frac{\partial^2 f}{\partial x^2} + k\frac{\partial^2 f}{\partial y \partial x}\right) + k\left(h\frac{\partial^2 f}{\partial x \partial y} + k\frac{\partial^2 f}{\partial y^2}\right) \\ &= h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2} \end{aligned}$$

Since F has its first two derivatives continuous on the interval $[0,1]$ it satisfies the hypothesis for the Taylor series expansion in a single variable with $n = 2$, so

$$\begin{aligned} F(1) &= F(0) + F'(0)(1-0) + F''(c)\frac{(1-0)^2}{2!} \qquad (\text{for } c \in [0,1]) \\ &= F(0) + F'(0) + \frac{1}{2}f''(c). \qquad \text{(B.1)} \end{aligned}$$

But $F(1) = f(a+h, b+k)$ and $F(0) = f(a,b)$ by definition. Also $F'(0) = hf_x(a,b) + kf_y(a,b)$ and $F''(c) = \left(h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}\right)$ evaulated at $(a+ch, b+ck)$.

Now substituting these expressions into (B.1) above

$$f(a+h, b+k) = f(a,b) + hf_x(a,b) + kf_y(a,b) + (\text{terms of higher order}),$$

and the ***tangent-plane approximation*** is

$$f(a+h, b+k) = f(a,b) + hf_x(a,b) + kf_y(a,b).$$