Appendix E

Maxima and minima of functions of two variables

E.1 Stationary points and the test for stationary points

The point (x_0, y_0) is a *stationary point* of the function $f(x, y)$ if

$$\frac{\partial f}{\partial x}(x_0, y_0) = 0,$$
$$\frac{\partial f}{\partial y}(x_0, y_0) = 0.$$

Lemma E.1 (Test for stationary points) *Let (x_0, y_0) is a stationary point of the function $f(x, y)$. Define*

$$A = \frac{\partial^2 f}{\partial x^2}(x_0, y_0),$$
$$B = \frac{\partial^2 f}{\partial y \partial x}(x, y),$$
$$C = \frac{\partial^2 f}{\partial y^2}(x_0, y_0).$$

Then

1. *If $AC - B^2 > 0$ and $A < 0$, the the stationary point is a maximum.*
2. *If $AC - B^2 > 0$ and $A > 0$, the the stationary point is a local minimum.*
3. *If $AC - B^2 < 0$ then the stationary point is a saddle point.*

Question E.1 *Suppose that the functions $f(x, y)$ and $g(x, y)$ are polynomial functions. When using Liapunov's direct method to investigate the stability of the trivial solution of the system*

$$\dot{x} = f(x, y),$$
$$\dot{y} = g(x, y)$$

a good guess is to try the function the function

$$V(x, y) = ax^2 + 2bxy + cy^2$$

(remark 3.11).

1. *Show that the trivial solution is a stationary point of the function V — is it the only one?*
2. *How does the classification of the trivial stationary point depend upon the parameter values a, b and c?*

E.2 Questions

1. Find the stationary points of the function

$$f(x, y) = x^3 + y^3 - 3x - 3y$$

   and classify them.

2. Find the stationary points of the function

$$f(x, y) = x^4 + 4x^2y^2 - 2x^2 + 2y^2 - 1$$

   and classify them.

3. Consider the function

$$f(x, y) = x^2 - 2xy + y^2.$$

   (a) Show that the function $f(x, y)$ has an infinite number of stationary points.
   (b) Show that the test for stationary points is inconclusive regarding their status.
   (c) Nevertheless, deduce from other reasoning that all of the stationary points are minima.

4. Find the stationary points of the following functions and determine their nature

   (a) $f(x, y) = x^3 + xy + y^2$
   (b) $f(x, y) = y^3 + 3x^2y - 3x^2 - 3y^2 + 2$
   (c) $f(x, y) = x^2 - y \sin(x)$

E.3 Things to do

Maple commands to visualise three-dimensional figures.