Appendix C

Scaling differential equations

C.1 What is scaling?

In this appendix we learn how to *scale*, or *non-dimensionalise*, a differential equation.

Consider a general differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f\left(x\right). \tag{C.1}$$

Suppose that we define a dimensionless variables x^* and t^* by

$$\begin{aligned} x^* &= \alpha x, \\ t^* &= \beta t. \end{aligned}$$

How do we obtain a differential equation for $\frac{\mathrm{d}x^*}{\mathrm{d}t^*}$ from equation (C.1)?

Note that if $x^* = \alpha x$ then $x = \frac{x^*}{\alpha}$. Thus

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x^*}{\alpha}\right), \\ &= \frac{1}{\alpha}\frac{\mathrm{d}x^*}{\mathrm{d}t}. \end{aligned}$$

In the scaled equations x is a function of t and in the unscaled equations x^* will be a function of t^*. Thus the last equation is really.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{\alpha}\frac{\mathrm{d}x^*\left(t^*\right)}{\mathrm{d}t}.$$

Note that x^* is a function of t^*. It is *not* a function of t. Thus using the chain rule we obtain

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{1}{\alpha}\frac{\mathrm{d}x^*}{\mathrm{d}t^*}\cdot\frac{\mathrm{d}t^*}{\mathrm{d}t}, \\ &= \frac{1}{\alpha}\frac{\mathrm{d}x^*}{\mathrm{d}t^*}\cdot\frac{\mathrm{d}\beta t}{\mathrm{d}t}, \\ &= \frac{\beta}{\alpha}\frac{\mathrm{d}x^*}{\mathrm{d}t^*}. \end{aligned}$$

Thus the differential equation C.1 becomes

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= f(x), \\
\frac{\beta}{\alpha}\frac{\mathrm{d}x^*}{\mathrm{d}t^*} &= f\left(\frac{x^*}{\alpha}\right), \\
\frac{\mathrm{d}x^*}{\mathrm{d}t^*} &= \frac{\alpha}{\beta} f\left(\frac{x^*}{\alpha}\right).
\end{aligned}$$

Example C.1 *Show that by introducing variables $S^* = S/K_s$, $X^* = X/(\alpha K_s)$ and $t^* = \mu_m t$ the system of differential equations*

$$\begin{aligned}
V\frac{dS}{dt} &= F(S_0 - S) - V\frac{\mu_m SX}{\alpha(K_s + S)} \\
V\frac{dX}{dt} &= F(X_0 - X) + V\frac{\mu_m SX}{K_s + S},
\end{aligned} \tag{C.2}$$

can be written in the scaled form

$$\begin{aligned}
\frac{dS^*}{dt^*} &= \frac{1}{\tau^*}(S_0^* - S^*) - \frac{X^*S^*}{1+S^*}, \\
\frac{dX^*}{dt^*} &= \frac{1}{\tau^*}(X_0^* - X^*) + \frac{X^*S^*}{1+S^*},
\end{aligned}$$

where the scaled parameters are: $S_0^* = S_0/K_s$, $X_0^* = X_0/(\alpha K_s)$, and $\tau_1^* = \mu_m \frac{V}{F}$.

The system of equations (C.2) is introduced in chapter 4 as a simple model for a bioreactor.

Solution Left for the student to do.

C.2 Questions

1. The Smith population model [22] is given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{r(K-x)}{K+ax}x,$$

   where a, r and K are all positive constants.

   What is the corresponding differential equation for the scaled variables $x^* = \frac{x}{K}$ and $t^* = rt$.

2. Consider the population model

$$\frac{\mathrm{d}x}{\mathrm{d}t} = rx\left(1 - \frac{x}{K}\right)\left(\frac{x}{K_0} - 1\right),$$

   where $r > 0$ and $0 < K_0 < K$.

   What is the corresponding differential equation for the scaled variables $x^* = \frac{x}{K}$ and $t^* = rt$.

3. The transmission dynamics of a disease in a population is represented by the equation

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \beta I\left(1 - \frac{I}{K}\right) - \alpha I,$$

   where I is the number of infected individuals in the population, β denotes the transmission coefficient ($\beta > 0$), K is the total population size ($K > 0$), and α is the recovery rate ($\alpha \geq 0$).

   (a) Show that the corresponding differential equation for the scaled variables $I^* = \frac{I}{K}$ and $t^* = \beta t$ can be written as

$$\frac{\mathrm{d}I^*}{\mathrm{d}t^*} = (1 - I^*)I^* - \lambda I^*$$

   and define the parameter λ in terms of the parameters α, β and K.

(b) Determine the steady-state solutions and their stability as a function of the parameter λ.

(c) (i) Draw a steady-state diagram showing how the steady-state solutions and their stability vary as a function of the parameter λ.

(ii) Identify the type of bifurcation that occurs in this model.

(iii) Explain the physical significance of the value of the parameter λ, relating your discussion to the value of this parameter at the bifurcation value.

4. The Nisbet & Gurney population model [19] is given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \left(r \exp\left[1 - \frac{x}{K}\right]\right) - d,$$

where the parameters d, r and K are all positive.

(a) Show that the corresponding differential equation for the scaled variables $x^* = \dfrac{x}{K}$ and $t^* = dt$ can be written as

$$\frac{\mathrm{d}x^*}{\mathrm{d}t^*} = \left(\lambda \exp\left[1 - x^*\right] - 1\right) x^*$$

and define the parameter λ in terms of the parameters d, r and K.

(b) Determine the steady-state solutions and their stability as a function of the parameter λ.

(c) (i) Draw a steady-state diagram showing how the steady-state solutions and their stability vary as a function of the parameter λ.

(ii) Identify the type of bifurcation that occurs in this model.

(iii) Explain the physical significance of the value of the parameter λ, relating your discussion to the value of this parameter at the bifurcation value.

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