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Student Number	______________
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University of Wollongong

School of Mathematics and Applied Statistics

MATH 971 — Non-Linear Differential Equations

Autumn Session Examination 2009

Time Allowed: 7 hours and 00 minutes

Number of Questions: 5.

Directions to Candidates

In the following instructions 'Maple' can be replaced by any mathematics package of your choice.

1. Each question is to be attempted.
2. The questions are *not* of equal value.
3. The examination paper is printed on both sides.
4. You are free to use Maple as much as you like to answer the questions.
5. A solution book is provided. Any Maple output should be attached to the solution book at the appropriate place.
6. WORKING (including all necessary reasoning) is to be shown for all solutions. If you used Maple to answer any part of a question you should attach a print-out with the commands you used and the generated output.
7. All notation is as used in lectures.

Examination Materials/Aids Allowed

All lecture notes, marked assignments, worked solutions to marked assignments and code from your assignments may be brought into the exam room.

Examination Materials/Aids to be supplied

None.

This examination paper must NOT be removed from the examination room.

Question 1. (**10 marks**)

For some organisms, finding a suitable mate may cause difficulties at low population densities. An appropriate model for the rate of change of population density under these circumstances is given by

$$x' = f(x), \qquad x(t=0) = x_0,$$
$$f(x) = rx^2\left(1 - \frac{x}{K}\right),$$

where $K > 0$, $r > 0$ and $x_0 > 0$. When such a population is subjected to constant yield harvesting the model becomes

$$x' = f(x) - h, \qquad x(t=0) = x_0, \tag{1}$$

where $h \geq 0$.

Part 1. Consider the case when there is no harvesting, i.e. $h = 0$.

(a) Find the steady-state solutions of equation (1) and determine their stability.

(b) Identify how the long-term behaviour of the model depends upon the choice of the initial condition (x_0).

(c) Under what circumstances is the solution $x(t)$ an increasing or decreasing function of time?

Part 2. Consider the case when there is harvesting, i.e. $h > 0$.

(a) Sketch the graph $y = f(x)$. By considering how the graph of $y = f(x) - h$ changes as h is increased from 0 identify the maximum sustainable value of the harvesting parameter, h_{cr}, on your sketch. Identify on your sketch the value of x, $\hat{x}$, that corresponds to the maximum sustainable value of the harvesting parameter.

(b) Determine an analytic expression for $\hat{x}$ in terms of r and K and the corresponding value for h_{cr}.

(c) Suppose that, for a particular species, $r = 1\,\text{year}^{-1}$ and that $K = 4$, measured in kilograms per unit area.

   (i) Describe how the long-term behaviour of the population depends upon the initial population size (x_0) if $h = 1.5$ (kilograms per unit area).

   (ii) Show a steady-state diagram for this model showing how the the population density x depends upon the value of the harvesting parameter h. Indicate the stability of the steady-state solutions, explaining how you determined their stability.

Question 2. (25 marks)

A mathematical model for a chemical process governed by autocatalytic kinetics with catalyst decay is given by

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t^*} = \frac{1-\alpha}{\tau^*} - \alpha\beta^2,$$
$$\frac{\mathrm{d}\beta}{\mathrm{d}t^*} = \frac{\beta_0-\beta}{\tau^*} + \alpha\beta^2 - \kappa_2\beta.$$

In these equations the primary bifurcation parameter is the dimensionless residence time $(\tau^*, \tau^* > 0)$. The secondary bifurcation parameters are the dimensionless inflow concentration of the catalyst, $\beta_0(\beta_0 > 0)$, and the dimensionless catalyst decay rate, $\kappa_2(\kappa_2 > 0)$. It can be shown that the steady-state value for α is bounded: $0 < \alpha < 1$.

(a) Show that the steady-state value for β is given by

$$\beta = \frac{1+\beta_0-\alpha}{1+\kappa_2\tau^*}.$$

(b) Hence, or otherwise, show that the singularity function for this model is given by

$$\mathcal{G}\left(\alpha,\tau^*,\kappa_2,\beta_0\right) = \left(1+\kappa_2\tau^*\right)^2\left(1-\alpha\right) - \tau^*\alpha\left(1+\beta_0-\alpha\right)^2 = 0.$$

(c) In the analysis from now on you can drop the star notation in your analysis.

(i) Show that the solution of the equations for the cusp singularity is given by

$$\alpha = \frac{3}{4},$$
$$\beta_0 = \frac{1}{8},$$

and the values of κ_2 and τ^* are related by the equation

$$0 = 64\kappa_2^2\tau^2 + \left(128\kappa_2 - 27\right)\tau + 64,$$

and that this is the only solution.

(ii) Deduce that there is a critical value of κ_2, $\kappa_{2,\mathrm{cr}}$, such that if $\kappa_2 > \kappa_{2,\mathrm{cr}}$ the cusp singularity does not occur.

(iii) Show that the only physically meaningful solution of the equations for the isola singularity is given by

$$\tau^* = \frac{1}{\kappa_2},$$
$$\beta_0^* = \left(1-\alpha\right)\left(2\alpha-1\right),$$
$$\kappa_2 = \alpha^3\left(1-\alpha\right).$$

(d) Construct a bifurcation diagram for the singularity function in the $\beta_0 - \kappa_2$ plane. In plotting the bifurcation diagram recall the restrictions on the parameter values given in the rubric for this question. Identify the location of any points on the bifurcation diagram where the non-degeneracy conditions fail.

(e) How many generic steady-state diagrams does this problem have?

Question 3. (2 marks)

Show that the equation

$$u'' + \phi(u)\, u' + \psi(u) = 0,$$

where the function $\phi(u)$ is always positive for all u (or is always negative for all u), has no periodic solutions.

Question 4. (10 marks)

In this question we consider a simple mathematical model for the spread of a *non-fatal* disease through a population. It is assumed that the total population size remains constant and that a vaccination program has been put into place.

The dynamics of the disease are given by the model

$$\begin{aligned} \frac{\mathrm{d}s}{\mathrm{d}t} &= \frac{b}{b+\gamma}(1-p-s) - R_0 s i, \quad && s(0) = s_0 \geq 0, \\ \frac{\mathrm{d}i}{\mathrm{d}t} &= (R_0 s - 1)\, i, && i(0) = r_0 > 0. \end{aligned} \tag{2}$$

In these equations s and i are the fraction of the population that are susceptible to the disease and that are infected by the disease respectively. The parameter R_0, $R_0 > 1$, is known as the basic reproductive ratio of the disease. The primary bifurcation parameter is p, $0 \leq p < 1$, which is the fraction of the population that is vaccinated. The parameters b and γ are strictly positive and are the *per-capita* birth rate and the the rate at which infected individuals recover from the disease respectively.

In analysing this model we are only interested in non-negative population fractions.

(a) Find the two steady-state solutions of system (2) and determine their stability as a function of p.

(b) Draw a steady-state diagram showing how the steady-state value of the infectives fraction (i^*) depends upon the value of the population that is vaccinated (p).

(c) What kind of bifurcation occurs in this model? Identify the value of (p, s^*, i^*) at which the bifurcation occurs.

(d) Hence, or otherwise, show that there is a critical value of p, p_{cr}, such that if $p < p_{\mathrm{cr}}$ the disease is eradicated from a population whilst if $p > p_{\mathrm{cr}}$ the disease is endemic in the population.

Question 5. (2 marks)

A particular form of the *Holling-Tanner* model is given by

$$\dot{x} = f(x, y) = x\left(1 - \frac{x}{7}\right) - \frac{6xy}{7 + 7x},$$
$$\dot{y} = g(x, y) = 0.2y\left(1 - \frac{Ny}{x}\right).$$

Consider the co-existent steady-state solution given by

$$(x^*, y^*) = \left(\hat{x}, \frac{\hat{x}}{N}\right),$$
$$\hat{x} = \frac{-6(1-N) + \sqrt{36(1-N)^2 + 28N^2}}{2N}.$$

Show that Hopf bifurcations occur at the parameters values

$$N_1 = \frac{1}{640}\left(327 + 11\sqrt{249}\right),$$
$$N_2 = \frac{1}{640}\left(327 - 11\sqrt{249}\right).$$

You may quote, without proving, relevant parts of the solution to question 1 of the week 7 assignment.

Do *not* determine if the bifurcations are sub- or super-critical.

END of EXAM