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First Name	______________
Student Number	______________
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University of Wollongong

School of Mathematics and Applied Statistics

MATH 971 — Non-Linear Differential Equations

Autumn Session Examination 2008

Time Allowed: 7 hours and 00 minutes

Number of Questions: 5.

Directions to Candidates

1. Each question is to be attempted.
2. The four questions are *not* of equal value.
3. The examination paper is printed on both sides.
4. You are free to use Maple as much as you like to answer the questions.
5. A solution book is provided. Any maple output should be attached to the solution book at the appropriate place.
6. WORKING (including all necessary reasoning) is to be shown for all solutions. If you used Maple to answer any part of a question you should attach a print-out with the commands you used and the generated output. You should add sufficient comments so that the examiner can understand your Maple code. In writing these comments assume that the examiner has minimal knowledge of Maple.
7. All notation is as used in lectures.

Examination Materials/Aids Allowed

All lecture notes, marked assignments, worked solutions to marked assignments and maple code from your assignments may be brought into the exam room.

Examination Materials/Aids to be supplied

None.

This examination paper must NOT be removed from the examination room.

Question 1. (**10 marks**)

Consider the *population* model

$$x' = f(x), \quad x(t=0) = x_0$$
$$f(x) = -rx\left(1 - \frac{x}{K_1}\right)\left(1 - \frac{x}{K_2}\right),$$

subject to constant yield harvesting

$$x' = f(x) - h, \quad x(t=0) = x_0. \tag{1}$$

where $0 < K_1 < K_2$, $r > 0$ and $h \geq 0$.

(a) Consider the case when there is no harvesting, i.e. $h = 0$.

(i) Find the steady-state solutions of equation (1) and determine their stability.

(ii) Identify how the long-term behaviour of the model depends upon the choice of the initial condition (x_0).

(b) Consider the case when there is harvesting, i.e. $h > 0$.

(i) Sketch the graph $y = f(x)$. By considering how the graph of $y = f(x) - h$ changes as h is increased from 0 identify the maximum sustainable value of the harvesting parameter, h_{cr}, on your sketch. Identify on your sketch the value of x, $\hat{x}$, that corresponds to the maximum sustainable value of the harvesting parameter.

(ii) Determine an analytic expression for x^* in terms of r, K_1 and K_2.

(iii) Suppose that, in a particular fishery, $r = 1\,\text{year}^{-1}$ and that $K_1 = 10^3$ and $K_2 = 10^4$, measured in kilograms. Determine the maximum sustainable yearly harvest (h_{cr}).

(iv) Suppose that, in a particular fishery, $r = 1\,\text{year}^{-1}$ and that $h = 4 \times 10^3$, $K_1 = 10^3$ and $K_2 = 10^4$, measured in kilograms. The current population size is estimated to be 1.5×10^3 (kilograms).

- Describe the long-term behaviour of the population.
- Given the estimated population size what is the maximum sustainable harvest yield?

Question 2. (13 marks)

Consider the singularity function

$$\mathcal{G} = x^2 + \lambda^4 + \alpha + \beta\lambda - 5\lambda^2, \tag{2}$$

where x is the state variable, λ is the primary bifurcation parameter and α & β are the secondary bifurcation parameters.

(a) Write down the conditions for an isola singularity to occur and solve . (Do not check the non-degeneracy conditions). Write the isola locus in the form $(\alpha(\lambda), \beta(\lambda))$.

(b) Identify the values of x, λ, α and β at any points where the non-degeneracy conditions for the isola singularity fail.

(c) Construct a bifurcation diagram for the singularity function in the $\alpha(x) - \beta(y)$ plane. Identify the location of any points on the bifurcation diagram where non-degeneracy conditions fail.

Remark. If the conditions for each singularity are expressed in the form $(\alpha(\lambda), \beta(\lambda))$ their locus can be drawn as a parametric plot.

(d) How many generic steady-state diagrams does equation (2) have?

Show an example of every generic steady-state diagram (plotting the state variable x as a function of the primary bifurcation parameter λ). For each steady-state diagram calculate the location of any limit point(s). Identify any limit point(s) on your steady-state diagram.

Question 3. (5 marks)

Determine the stability of the origin in the system

$$\begin{aligned} \dot{x} &= -y - x^3 - xy^2, \\ \dot{y} &= x - y^3 - yx^2. \end{aligned} \tag{3}$$

If the origin is stable, is it globally asymptotically stable?

Question 4. (**10 marks**)

In this question we consider a simple mathematical model for the spread of a *non-fatal* disease through a population. It is assumed that the total population size remains constant.

The dynamics of the disease are given by the model

$$\begin{aligned} \frac{\mathrm{d}s}{\mathrm{d}\tau} &= \frac{b}{b+\gamma}\left(1-s\right) - R_0 s i, \quad & s\left(0\right) = s_0 \geq 0, \\ \frac{\mathrm{d}i}{\mathrm{d}\tau} &= \left(R_0 s - 1\right) i, & i\left(0\right) = i_0 > 0. \end{aligned} \tag{4}$$

In these equations s and i are the fraction of the population that are susceptible to the disease and that are infected by the disease respectively. The parameter R_0, $R_0 > 0$, is known as the basic reproductive ratio of the disease. The parameters b and γ are strictly positive and are the *per-capita* birth rate and the the rate at which infected individuals recover from the disease respectively.

In analysing this model we are only interested in non-negative population fractions.

(a) Find the two steady-state solutions of system (4) and determine their stability as a function of R_0.

(b) Draw a steady-state diagram showing how the steady-state value of the infectives fraction (i^*) depends upon the value of R_0.

(c) Hence, or otherwise, show that there is a critical value of R_0, $R_{0,\mathrm{cr}}$, such that if $R < R_{0,\mathrm{cr}}$ the disease is eradicated from a population whilst if $R > R_{0,\mathrm{cr}}$ the disease is endemic in the population.

Question 5. 6 marks

Check that the following system has an equilibrium point that exhibits the Hopf bifurcation at some value of α. Compute the first Lyapunov coefficient and determine if the Hopf bifurcation is subcritical or supercritical.

$$\ddot{y} - \left(\alpha - y^2\right)\dot{y} + y = 0.$$

END of EXAM