School of Mathematics & Applied Statistics

MATH971: Applied Non-Linear Differential Equations

Assignment Week 13

Autumn 2008

Student Name: ________________________ *Student Number:* ____________

FULL WORKING is to be shown for all solutions.
Untidy or badly set out work will not be marked and will be recorded as unsatisfactory.
This assignment is to be handed in during the Tuesday of stuvac

1. Analysis of a mathematical model for a chemostat with a variable yield coefficient [4] shows that the values of the residence time at which Hopf bifurcations occur correspond to the roots of

$$\mathcal{H}\left(\tau^*\right) = -S_0^* {\tau^*}^3 + 3S_0^* {\tau^*}^2 - \left[3S_0^* + 1 + \beta^*\left(1 - S_0^*\right)\right]\tau^* - \left(\beta^* - 1\right)\left(1 + S_0^*\right) = 0, \tag{1}$$

subject to constraint that $\tau^* > 1 + \frac{1}{S_0^*}$. In this equation S_0^* is the dimensionless substrate concentration in the feed, β^* is the dimensionless variable yield coefficient and τ^* is the dimensionless residence time.

(a) Show that when $\beta^* = 5.25$ there is a double Hopf bifurcation at the parameter values

$$\left(S_{0,\text{cr}}^*, \tau^*\right) = (3.9097, 2.103)\,.$$

(b) With the stated value of β^* do Hopf bifurcations occur in the model for $S_0^* > S_{0,\text{cr}}^*$ or for $S_0^* > S_{0,\text{cr}}^*$?

(c) Construct a diagram showing how the critical value of $S_0^* > S_{0,\text{cr}}^*$ varies as a function of β^* with $0 < \beta^* \leq 20$.

2. The normal form of the Bautin bifurcation is given by the system

$$\begin{aligned} \dot{\rho} &= \rho\left(\beta_1 + \beta_2\rho^2 - \rho^4\right), \\ \dot{\theta} &= 1. \end{aligned} \tag{2}$$

Analyse the steady-state solutions of equation (2) in the $\beta_1 - \beta_2$ plane.

(a) Identify parameter regions in which there are zero, one and two limit cycles.

(b) Determine the stability of the limit-cycles identified in your answer to the previous question.

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Tutorial Class: __________ *Date Submitted:* ____________ *Tutor Initials:* ____________

3. The following predator-prey system is a generalisation of the Volterra equations [2]

$$\dot{x} = \frac{x^2(1-x)}{n+x} - xy,$$
$$\dot{y} = -\gamma y(m-x),$$

where m, n, and γ are positive parameters. (See also [1] for more information about this model).

(a) Derive the equation for the Hopf bifurcation curve in the system and show that it is independent of γ.

(b) Compute the first Liapunov coefficient along the Hopf curve and show that it vanishes at a Bautin point when

$$(m, n) = \left(\frac{1}{4}, \frac{1}{8}\right).$$

[3, exercise 3 on page 324]

References

[1] A. Bazykin. *Mathematical Biophysics of Interacting Populations.* Nauka, Moscow, 1955. In Russian.

[2] A. Bazykin and A. Khibnik. On sharp excitation of self-oscillations in a Volterra-type model. *Biophysika*, 26:851–853, 1981. In Russian.

[3] Y.A. Kuznetsov. *Elements of Applied Bifurcation Theory.* Applied Mathematical Sciences 112. Springer-Verlag, 1st edition, 1995.

[4] M.I. Nelson and H.S. Sidhu. Analysis of a chemostat model with variable yield coefficient. *Journal of Mathematical Chemistry*, 38(4):605–615, 2005.