School of Mathematics & Applied Statistics

MATH971: Applied Non-Linear Differential Equations

Assignment Week 8

Autumn 2008

Student Name: ______________________________ *Student Number:* ____________

FULL WORKING is to be shown for all solutions.
Untidy or badly set out work will not be marked and will be recorded as unsatisfactory.
This assignment is to be handed in during the examples class in Week 9.

1. A particular form of the *Holling-Tanner* model is given by

$$\dot{x} = x\left(1 - \frac{x}{7}\right) - \frac{6xy}{7 + 7x},$$
$$\dot{y} = 0.2y\left(1 - \frac{Ny}{x}\right).$$

   (a) Find the steady-state solutions of this model, and determine their stability, as a function of the parameter.
   (b) Draw steady-state diagrams showing how the values for x and y vary as a function of N.
   (c) What does the Holling-Tanner model model?

2. Consider the system

$$\dot{x} = -y - x^3,$$
$$\dot{y} = x - y^3.$$

   (i) What does a linear stability analysis tell you about the stability of the origin?
   (ii) Investigate the stability of the origin using the function

$$V = ax^2 + by^2,$$

   where the values for a and b are to be established.

3. For each of the following systems:

$$\begin{aligned}
(a)\quad & \dot{z_1} = -2z_1 + z_2^3, & \dot{z_2} = z_1 - z_2 + z_1 z_2^2;\\
(b)\quad & \dot{z_1} = z_1 - z_2 + z_1^2 \sin z_2, & \dot{z_2} = -2z_2 + z_1^3;\\
(c)\quad & \dot{z_1} = 3z_1 + 2z_2 + z_2^2, & \dot{z_2} = -10z_1 - 5z_2 - z_1^2 z_2,
\end{aligned}$$

   use a suitable Liapunov function $V(z_1, z_2)$ to determine the stability of the zero solution. In each case start try

$$V = az_1^2 + 2bz_1z_2 + cz_2^2$$

   and determine the coefficients a, b and c to give V and $\dot{V}$ suitable properties. Compare with the results obtained using the principle of linearised stability.

   Hint. Choose a, b and c so that $\dot{V}$ is approximately $\pm\left(z_1^2 + z_2^2\right)$, with the negative sign for a stable steady-state solution and a positive sign for an unstable steady-state solution.

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Tutorial Class: ________ *Date Submitted:* ____________ *Tutor Initials:* ____________

4. Spicer (1955) analysed a model for the concentration of a nutrient (S) and microorganism feeding upon the nutrient (X) in a continuously stirred tank reactor which can be written as follows.

$$\begin{aligned} \frac{\mathrm{d}S}{\mathrm{d}t} &= \frac{1}{\tau}\left(S_0 - S\right) - \frac{\mu_{\max}}{\alpha} \cdot XS, \\ \frac{\mathrm{d}X}{\mathrm{d}t} &= \mu_{\max} XS - \frac{1}{\tau}X. \end{aligned} \tag{1}$$

In this model: S_0 is the concentration of nutrient in the incoming medium; t is time; α is known as the yield constant; μ_{max} is the growth rate of microorganisms feeding upon nutrients; τ is the residence time, which is the main parameter that is experimentally controllable. All constants in this model are non-negative.

(a) Find the steady-state solutions of system (1) and determine their stability as a function of the residence time.

(b) We are only interested in steady-state solutions in which the corresponding values for S and X are non-negative. Does this impose any restrictions on the parameters?