School of Mathematics & Applied Statistics

MATH971: Applied Non-Linear Differential Equations

Assignment Week 2

Autumn

Student Name: ______________________________ *Student Number:* ______________

FULL WORKING is to be shown for all solutions.
Untidy or badly set out work will not be marked and will be recorded as unsatisfactory.
This assignment is to be handed in during the examples class in Week 3.

1. Consider the population model (Smith 1963)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{r(K-x)}{K+ax}x, \quad x(0) = x_0.$$

(The parameters $a, r, \& K$ are strictly positive. The parameter x_0 is strictly non-negative).

   (a) Sketch $\frac{\mathrm{d}x}{\mathrm{d}t}$ as a function of x. Hence determine how the long-term dynamics of the model depends upon the initial value x_0.

   (b) Sketch the solution $x(t)$. How does the qualitative form of your sketch depend upon the initial value x_0?

   (c) Determine the stability of the steady-state solutions of this model.

2. For which initial values $y(0)$ does the solution $y(t)$ of the differential equation

$$y' = y\left(e^{-y} - 2y\right)$$

approach zero as $t \to \infty$? (Note that is *not* a population model so that $y(0)$ may be negative). [Brauer & Castillo-Chávez]

3. Suppose that x^* is a steady-state solution of the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x)$$

such that $f'(x^*) = 0$.

   (a) Show that a stability analysis leads to the equation

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} = \frac{1}{2!}f''(x^*)\,\xi^2.$$

   (b) How does the stability of the steady-state solution depend upon the value of $f''(x^*)$?
   HINT. Sketch the function $\frac{1}{2!}f''(x^*)\,\xi^2$.

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Tutorial Class: __________ *Date Submitted:* ______________ *Tutor Initials:* ______________

4. Suppose that x^* is a steady-state solution of the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x)$$

such that $f'(x^*) = f''(x^*) = 0$.

By carrying out a suitable Taylor series expansion of the function $f(x)$ determine the stability of the steady-state solution.

5. The spruce budworm model is given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = rx\left(1 - \frac{x}{q}\right) - \frac{x^2}{1+x^2},$$

with $r > 0$ and $q > 0$.

(a) Show that the steady-state(s) of the spruce budworm model are given by $x_1 = 0$ and the solutions of the algebraic equation

$$\mathcal{G} = r\left(1+x^2\right)\left(1 - \frac{x}{q}\right) - x = 0.$$

(b) Determine the stability of the trivial solution $x_1 = 0$.

(c) Suppose that the value of q is such that the steady-state diagram of the function $\mathcal{G}$ is given by figure 1.8 of the lecture notes.

(i) Show that when the function $\mathcal{G}$ has a single steady-state it is stable.

(ii) Show that when the function $\mathcal{G}$ has three steady-states the lower and upper solutions are stable and the middle-solution is unstable.

(iii) Hence sketch the full steady-state diagram when $q = 20$, indicating stability. Don't forget to include the solution $x_1 = 0$.

Hint Graphical techniques make the stability analysis easy!

(d) Suppose that the value of q is such that the steady-state diagram for the function $\mathcal{G}$ is given by figure 1.9 of the lecture notes.

(i) Show the the unique steady-state solution of the the function $\mathcal{G}$ is stable.

(ii) Hence sketch the *full* steady-state diagram when $q = 3$, indicating stability.

(e) By generating steady-state diagrams for different values of q find a critical value of q, q_{cr}, (to one decimal place) at which the features of the steady-state diagram changes from the type shown in figure 1.8 of the notes to that shown in figure 1.9.

(f) Try to explain how the steady-state diagram changes during the transition from one type to the other.