

School of Mathematics & Applied Statistics  
**MATH971: Applied Non-Linear Differential Equations**  
**Assignment Week 12**  
**Autumn 2010**

*Student Name:* \_\_\_\_\_ *Student Number:* \_\_\_\_\_

FULL WORKING is to be shown for all solutions.

Untidy or badly set out work will not be marked and will be recorded as unsatisfactory.

This assignment is to be handed in during the examples class in Week 14.

The assignment that you hand in *must* include a cover page. On the cover-page you should briefly answer the following questions

- (a) What topic did you believe was the most important in the assignment?
- (b) Why do you believe that is the most important topic?
- (c) What problems did you have with the assignment, if any?

You should answer each question with a complete sentence.

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School of Mathematics & Applied Statistics **MATH971: Applied Non-Linear Differential Equations**  
**Assignment Week 14**  
**Autumn 2010 Submission Receipt**

*Student Name:* \_\_\_\_\_ *Student Number:* \_\_\_\_\_

*Tutorial Class:* \_\_\_\_\_ *Date Submitted:* \_\_\_\_\_ *Tutor Initials:* \_\_\_\_\_

1. Analysis of a mathematical model for a chemostat with a variable yield coefficient [1] shows that the values of the residence time at which Hopf bifurcations occur correspond to the roots of

$$\mathcal{H}(\tau^*) = -S_0^* \tau^{*3} + 3S_0^* \tau^{*2} - [3S_0^* + 1 + \beta^* (1 - S_0^*)] \tau^* - (\beta^* - 1)(1 + S_0^*) = 0, \quad (1)$$

subject to constraint that  $\tau^* > 1 + \frac{1}{S_0^*}$ . In this equation  $S_0^*$  is the dimensionless substrate concentration in the feed ( $S_0^* > 0$ ),  $\beta^*$  is the dimensionless variable yield coefficient ( $\beta^* > 0$ ) and  $\tau^*$  is the dimensionless residence time ( $\tau^* > 0$ ).

- (a) Draw the Hopf bifurcation unfolding curve in the  $S_0^* - \tau^*$  plane when  $\beta^* = 5.25$ .  
 (b) Hence show that when  $\beta^* = 5.25$  there is a double Hopf bifurcation at the parameter values

$$(S_{0,\text{cr}}^*, \tau^*) = (3.9097, 2.103).$$

- (c) With the stated value of  $\beta^*$  do Hopf bifurcations occur in the model for  $S_0^* > S_{0,\text{cr}}^*$  or for  $S_0^* < S_{0,\text{cr}}^*$ ?  
 (d) Construct a diagram showing how the critical value of the dimensionless substrate concentration  $S_{0,\text{cr}}^*$ , varies as a function of the dimensionless variable yield coefficient  $\beta^*$  with  $0 < \beta^* \leq 20$ .  
 On your diagram marks the regions in which there are zero and two Hopf bifurcation points respectively.

2. The normal form of the Bautin bifurcation is given by the system

$$\begin{aligned} \dot{\rho} &= \rho (\beta_1 + \beta_2 \rho^2 - \rho^4), \\ \dot{\theta} &= 1. \end{aligned} \quad (2)$$

Analyse the steady-state solutions of equation (2) in the  $\beta_1 - \beta_2$  plane.

- (a) Identify parameter regions in which there are zero, one and two limit cycles.  
 (b) Hence, or otherwise, draw a figure in the  $\beta_1 - \beta_2$  plane showing how the number of limit cycles depends upon the values of  $\beta_1$  and  $\beta_2$ .  
 (c) Determine the stability of the limit-cycles identified in your answer to the previous question.
3. In this question we return to the model for cubic auto-catalysis.

$$\begin{aligned} \frac{d\alpha}{dt^*} &= \frac{1 - \alpha}{\tau^*} - \alpha\beta^2, \\ \frac{d\beta}{dt^*} &= \frac{\beta_0 - \beta}{\tau^*} + \alpha\beta^2 - \kappa_2\beta. \end{aligned}$$

In these equations the primary bifurcation parameter is the dimensionless residence time ( $\tau^*$ ,  $\tau^* > 0$ ). The secondary bifurcation parameters are the dimensionless inflow concentration of the autocatalyst ( $\beta_0$ ,  $\beta_0 \geq 0$ ) and the dimensionless decay rate of the autocatalyst ( $\kappa_2$ ,  $\kappa_2 > 0$ ).

- (a) Show that the condition for the double-zero eigenvalue bifurcation can be written in the form

$$\begin{aligned} \beta_0 &= f(\alpha), \\ \kappa_2 &= g(\alpha). \end{aligned}$$

- (b) Plot the double-zero eigenvalue locus in the  $\kappa_2 - \beta$  plan. Is the region in which a Hopf bifurcation exists the interior or exterior of this locus? Carefully justify your answer.

## References

- [1] M.I. Nelson and H.S. Sidhu. Analysis of a chemostat model with variable yield coefficient. *Journal of Mathematical Chemistry*, 38(4):605–615, 2005.