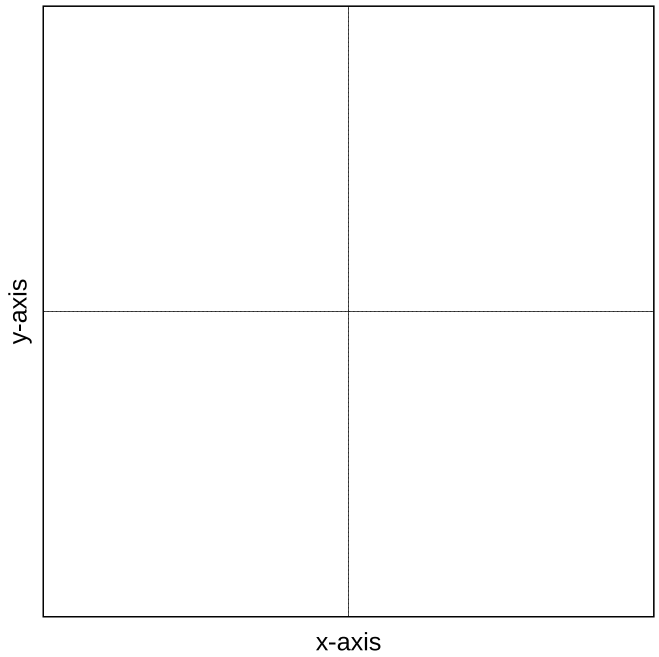
POLAR COORDINATES and POLAR CURVES

4.1 Polar Coordinates

Introduction In this chapter we introduce polar co-ordinates. We will define polar coordinates and look at their relationship with Cartesian coordinates. One major distinction between the two is that while each point in the plane has only one pair of Cartesian coordinates, it has infinitely many pairs of polar coordinates. We will also look at graphing polar coordinates.



Example Plot the point $P(2, 2)$.

- Find the distance, r, from P to the origin O.
- Find the angle, θ, that OP makes with the positive x-axis.

Distance

$$r =$$
$$=$$

Angle

$$\tan\theta =$$
$$=$$
$$\therefore \theta =$$
$$=$$

$\sqrt{2^2+2^2}$

$2\sqrt{2}$

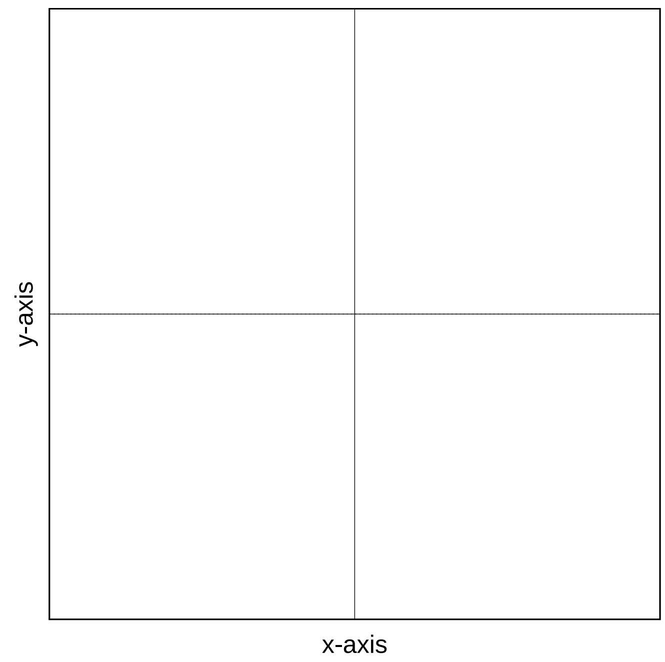
$\frac{2}{2}$

1

$\tan^{-1} 1$

$\frac{\pi}{4}$

Example



Find the **co**-ordinates of a point Q that is 2 units from the origin O and where OQ makes an angle of $\pi/3$ radians with the positive x-axis.

x-coordinate

$$\cos\frac{\pi}{3} = \frac{x}{2}$$

$$\therefore \quad x = 2\cos\frac{\pi}{3}$$

$$= 2 \times \frac{1}{2} = 1$$

y-coordinate

$$\sin\frac{\pi}{3} = \frac{y}{2}$$

$$\therefore \quad y = 2\sin\frac{\pi}{3}$$

$$= 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$$

The coordinates of the point Q are therefore $\left(1, \sqrt{3}\right)$.

4.1.1 Description of polar coordinates

In Cartesian coordinates we describe the position of a point using the notation (x, y) where

x is the horizontal distance from the origin, and

y is the vertical distance from the origin.

It is possible to determine the position of a point by coordinates other than Cartesian.

In the previous examples we used

r the distance of the point from the origin, and

θ the angle between the positive x-axis and the line from the origin to the point.

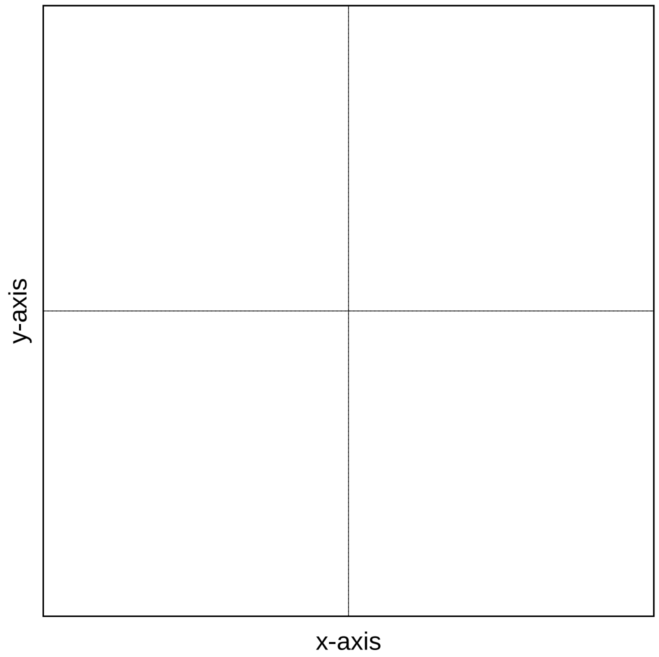
These coordinates are known as polar **c**o-ordinates (in two dimensions).

Polar coordinates are represented by (r, θ) where r is the distance from the origin and θ is the angle with the positive x-axis (in radians).

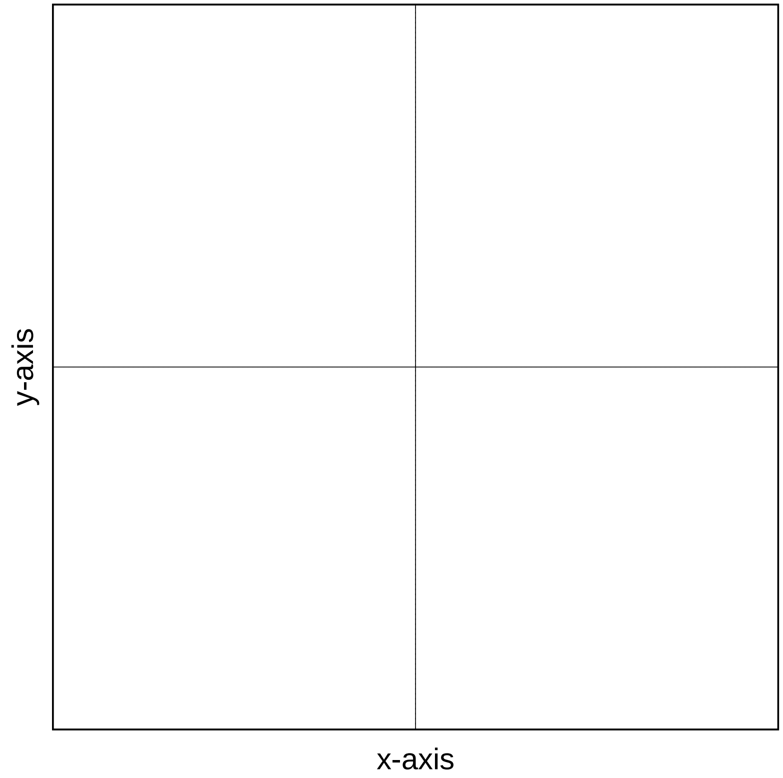
Plot the points whose polar coordinates are

$$A\left(3, \frac{\pi}{6}\right); \quad B\left(2, \frac{7\pi}{4}\right);$$

$$C\left(\frac{3}{2}, \frac{5\pi}{2}\right); \quad D\left(\frac{\pi}{2}, 1\right)$$



$C\left(\frac{3}{2}, \frac{5\pi}{2}\right)$ and $D\left(\frac{\pi}{2}, 1\right)$



Just as in Cartesian coordinates where we can have positive and negative values of x and y, we can have positive and negative values of r and θ in polar coordinates.

The conventions are:

θ is positive in the anti-clockwise direction and negative in the clockwise direction.

r is positive going forward from the origin along the side of the angle and negative going backwards from the origin.

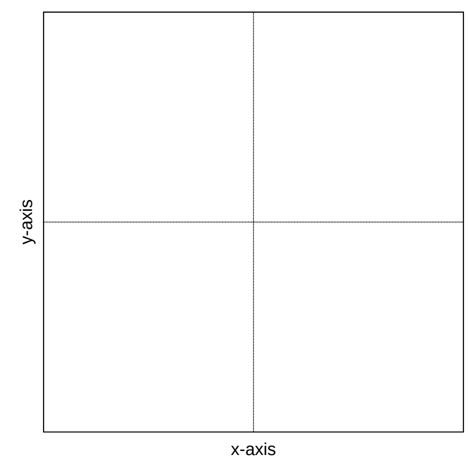
This means that there are many representations for one particular point.

Example Plot the points with polar coordinates

$$E\left(2, \frac{-3\pi}{4}\right); \quad F\left(-4, \frac{3\pi}{4}\right);$$

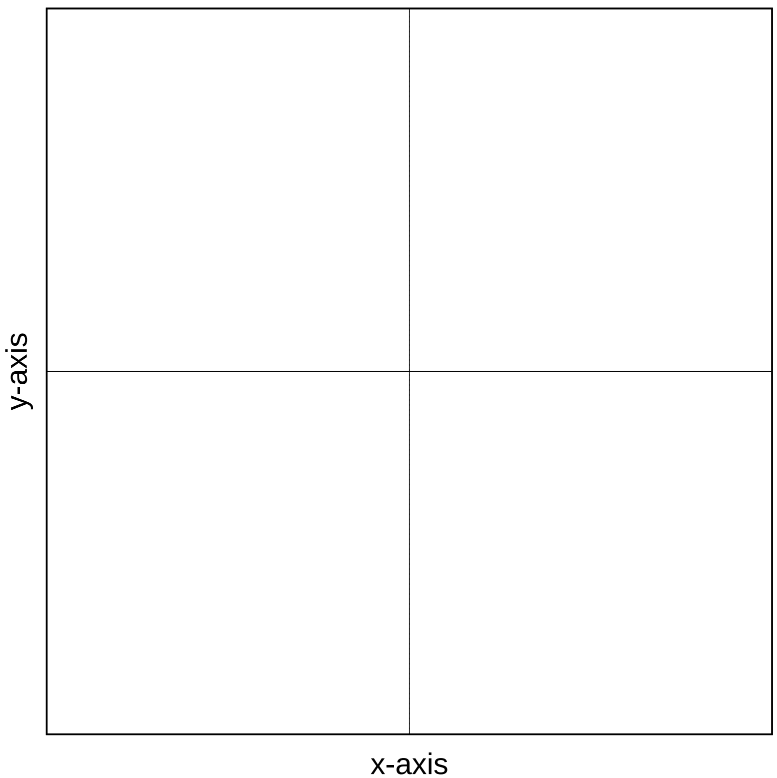
$$G\left(3, \frac{5\pi}{4}\right); \quad H\left(-3, \frac{\pi}{4}\right);$$

$$I\left(3, \frac{-3\pi}{4}\right); \quad J\left(3, \frac{3\pi}{4}\right).$$

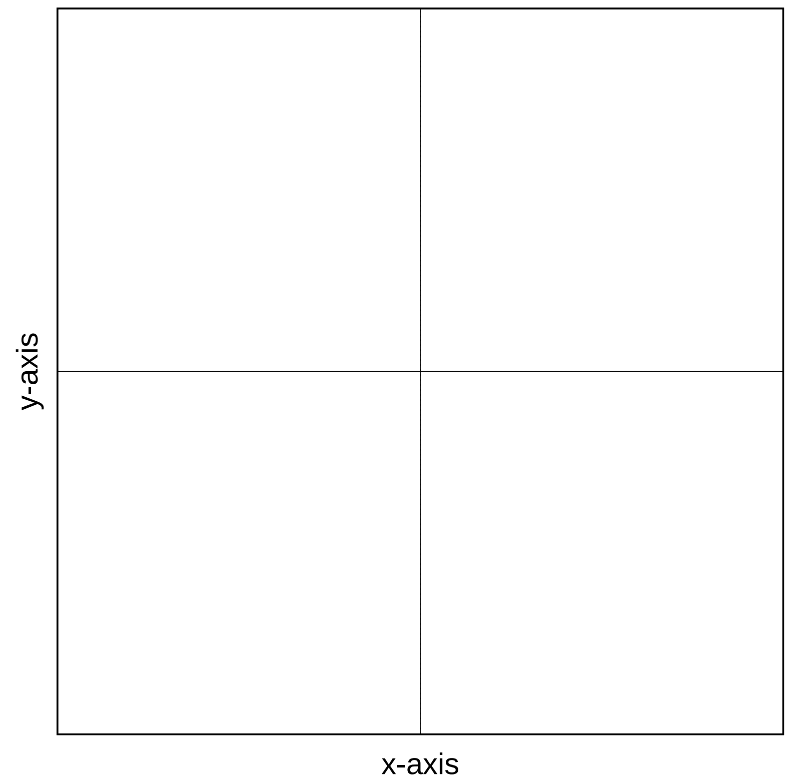


$$E\left(2, \frac{-3\pi}{4}\right); \quad F\left(-4, \frac{3\pi}{4}\right)$$

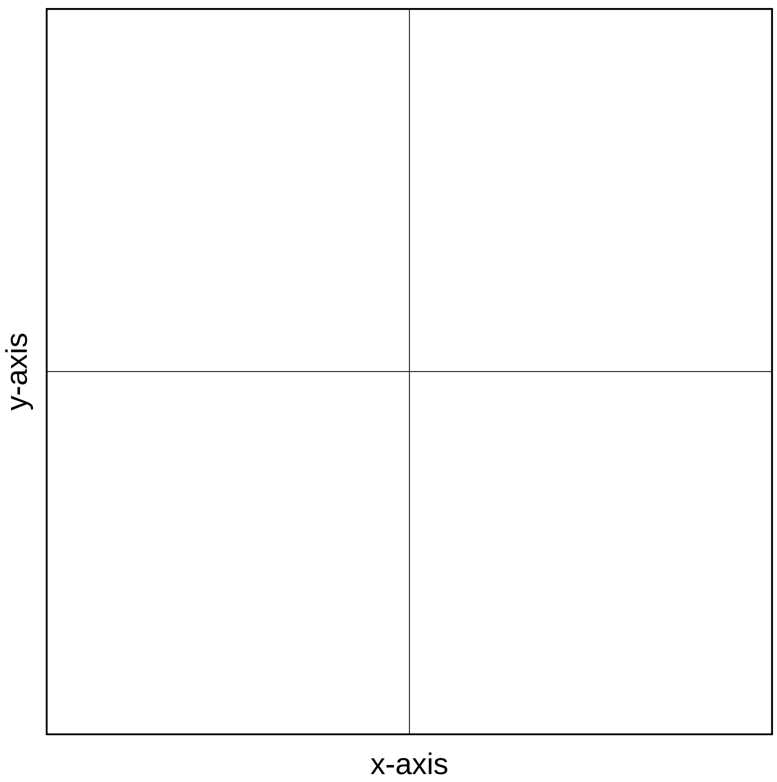
$$G\left(3, \frac{5\pi}{4}\right); \quad H\left(-3, \frac{\pi}{4}\right)$$



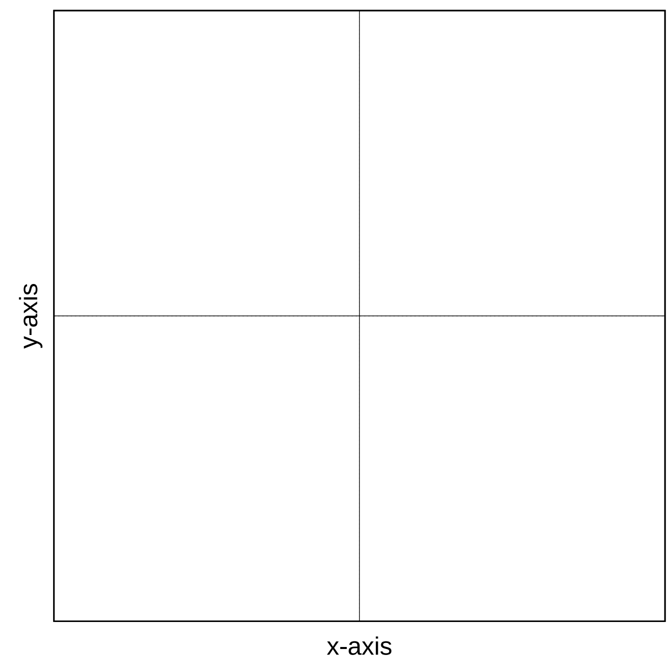
$$I\left(3, \frac{-3\pi}{4}\right); \quad J\left(3, \frac{13\pi}{4}\right).$$



Find two other polar coordinate representations for the point $\left(1, -\frac{\pi}{3}\right)$.



4.1.2 Relationship between Polar and Cartesian Coordinates

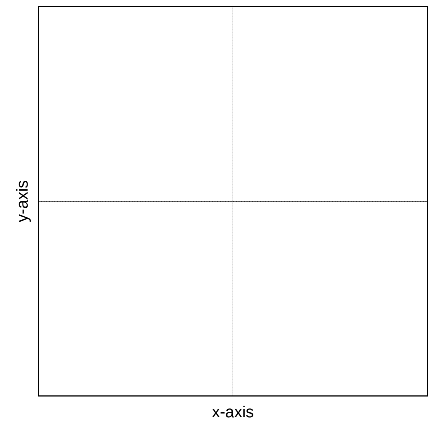


From the diagram

$$\cos\theta = \frac{x}{r} \quad \Rightarrow \quad x = r\cos\theta$$
$$\sin\theta = \frac{y}{r} \quad \Rightarrow \quad y = r\sin\theta$$

Also

$$\tan\theta = \frac{y}{x} \quad \Rightarrow \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$
$$r^2 = x^2 + y^2 \quad \Rightarrow \quad r = \sqrt{x^2 + y^2}$$



Example Convert $\left(5, \frac{3\pi}{4}\right)$ from polar coordinates to Cartesian.

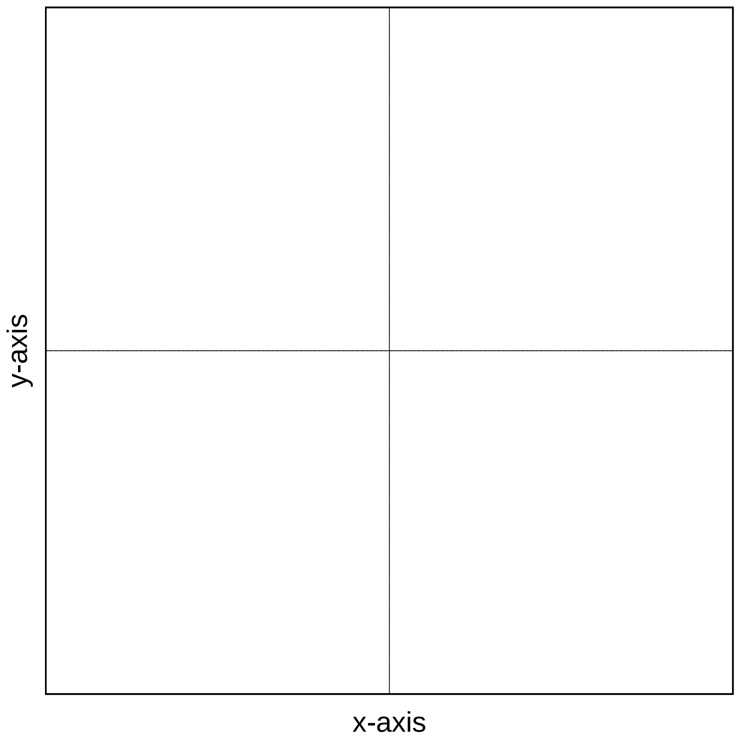
We have $r = 5$ and $\theta = \frac{3\pi}{4}$.

$$\begin{aligned} \therefore \quad x &= r\cos\theta & y &= r\sin\theta \\ &= 5\cos\frac{3\pi}{4} & &= 5\sin\frac{3\pi}{4} \\ &= 5\left(\frac{-1}{\sqrt{2}}\right) & &= 5\cdot\frac{1}{\sqrt{2}} \\ &= \frac{-5}{\sqrt{2}} & &= \frac{5}{\sqrt{2}} \end{aligned}$$

$\therefore \quad \left(5, \frac{3\pi}{4}\right)$ converts to $\left(\frac{-5}{\sqrt{2}}, \frac{5}{\sqrt{2}}\right)$.

(Note: You *should* check that both points are in the same quadrant!)

Example Convert $(1, -\sqrt{3})$ from Cartesian coordinates to polar.



We have $x = 1$ and $y = -\sqrt{3}$

Distance

$$r =$$

$$=$$

$$=$$

$$=$$

Angle

$$\tan\theta =$$

$$\Rightarrow \theta =$$

$$=$$

$\therefore \quad (1, -\sqrt{3})$ converts to $\left(2, \dfrac{-\pi}{3}\right)$.

$$\sqrt{x^2 + y^2}$$

$$\sqrt{(1)^2 + \left(-\sqrt{3}\right)^2}$$

$$\sqrt{1 + 3}$$

$$2$$

$$\frac{y}{x}$$

$$\tan^{-1}\left(\frac{-\sqrt{3}}{1}\right)$$

$$\frac{-\pi}{3}$$

Note: When converting from Cartesian coordinates to polar coordinates you should *always* draw a quick sketch to make sure you know in which quadrant the point lies (and hence choice of θ).

4.1.3 Revision Questions

The folowing questions are about the key ideas in this section.

1. The point P has polar coordinates (r, θ). What do the symbols r and θ represent?
2. The point P has polar coordinates (r, θ). What are the Cartesian coordinates of this point?
3. The point P has polar coordinates (r, θ). How many other ways are there to represent P using polar coordinates? If we convert to Cartesian coordinates how many ways are there to represent P?
4. Let R and S be the points (r, θ) and $(r, -\theta)$. Where is the point S relative to the point R? (Hint. Draw a graph).

5. Let R and S be the points (r, θ) and $(-r, \theta)$. Where is the point S relative to the point R? (Hint. Draw a graph).

6. You are asked to convert a point with polar coordinates to Cartesian coordinates (or vice-versa). What is the first thing that you should do?

7. **Exercise** 4.1.4.