3.3 Methods of integration

3.3.1 Integrals by inspection

Integration is the reverse of differentiation. Our standard derivatives provide us with a source of standard integrals that we can evaluate on sight.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^{n+1}}{n+1}\right) = x^n$$

$$\therefore \int x^n \mathrm{d}x = \frac{x^{n+1}}{n+1} + c, \quad n \neq -1$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(ax) = a$$

$$\therefore \int a \mathrm{d}x = ax + c$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = \cos x$$

$$\therefore \int \cos x \mathrm{d}x = \sin x + c$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cos x) = -\sin x$$

$$\therefore \int \sin x \mathrm{d}x = -\cos x + c$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{x}\right) = e^{x}$$

$$\therefore \int e^{x} = e^{x} + c$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln x\right) = \frac{1}{x}$$

$$\therefore \int \frac{1}{x}\mathrm{d}x = \ln x + c$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sinh x\right) = \cosh x$$

$$\therefore \int \cosh x \mathrm{d}x = \sinh x + c$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\cosh x\right) = \sinh x$$

$$\therefore \int \sinh x \mathrm{d}x = \cosh x + c$$

Example Evaluate

1. $\displaystyle\int \left(3x^{6} - 2x^{2} + 7x + 1\right) \mathrm{d}x$
2. $\displaystyle\int \left(\frac{4}{\sqrt{x}} + \frac{1}{2x^{2}}\right) \mathrm{d}x$
3. $\displaystyle\int_{0}^{\pi} \left(3e^{x} - 4\cos x\right) \mathrm{d}x$
4. $\displaystyle\int_{1}^{2} \left(x^{-3} + \frac{1}{x} - \sinh x\right) \mathrm{d}x$

Answers

1. $\dfrac{3x^{7}}{7} - \dfrac{2x^{3}}{3} + \dfrac{7x^{2}}{2} + x + c$
2. $8\sqrt{x} - \dfrac{1}{2x} + c$
3. $3\left(e^{\pi} - 1\right)$
4. $\dfrac{3}{8} + \ln 2 - \cosh 2 + \cosh 1$

3.3.2 Simplifying Integrals

Sometimes an integrand can be simplified before evaluating the integral.

Examples

$$\int \frac{t^2 - 2t^4}{t^4} \mathrm{d}t =$$

$$=$$

$$=$$

$$\int \left(2 + x^2\right)^2 \mathrm{d}x =$$

$$=$$

$$\int \left(\frac{1}{t^2} - 2\right) \mathrm{d}t$$

$$\int \left(t^{-2} - 2\right) \mathrm{d}t$$

$$-1t^{-1} - 2t + c$$

$$\int \left(4 + x^4 + 4x^2\right) \mathrm{d}x$$

$$4x + \frac{x^5}{5} + \frac{4}{3}x^3 + c$$

$$\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{\cos x} \mathrm{d}x =$$

$$=$$

$$=$$

$$=$$

$$=$$

$$=$$

$$\int \frac{2\sin x\cos x}{\cos x}\,\mathrm{d}x$$

$$\int 2\sin x\,\mathrm{d}x$$

$$2\left[-\cos x\right]_0^{\pi/2}$$

$$2\left[-\cos\left(\frac{\pi}{2}\right)+\cos(0)\right]$$

$$2$$

Exercise Evaluate the following integrals.

1. $\displaystyle\int \frac{1}{\sec x}\mathrm{d}x$
2. $\displaystyle\int_0^{\pi} \left(\cos^2 x+\sin^2 x\right)\mathrm{d}x$
3. $\displaystyle\int \frac{x^2\sin x+2\sin x}{x^2+2}\mathrm{d}x$
4. $\displaystyle\int x^{1/3}(2-x)\,\mathrm{d}x$
5. $\displaystyle\int_0^1 \frac{x^2-1}{x+1}\mathrm{d}x$
6. $\displaystyle\int \sqrt[3]{x^2}\mathrm{d}x$

$c =$ arbitrary constant

$\sin(x) + c$

π

$-\cos(x) + c$

$\frac{3}{2}x^{4/3} - \frac{3}{7}x^{7/3} + c$

$-\frac{1}{2}$

$\frac{3}{5}x^{5/3} + c$

3.3.3 Using Integral Tables

- The integration tables at the back of the course book contain 40 common integrals.
- The same tables will be included in the final exam for MATH 141.
- You should work out which integrals are in the tables and where they are.

Example Evaluate $\int \frac{\mathrm{d}x}{x\sqrt{2x+4}}$

Solution Use formula [17].

$$\int \frac{1}{x\sqrt{ax+b}}\mathrm{d}x = \frac{1}{\sqrt{b}} \ln\left|\frac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}}\right| + c$$

In our question

$$a = \_ \qquad 2$$
$$b = \_ \qquad 4$$

Therefore we have

$$\int \frac{1}{x\sqrt{ax+b}}\mathrm{d}x =$$

$$\frac{1}{\sqrt{4}}\ln\left|\frac{\sqrt{2x+4}-\sqrt{4}}{\sqrt{2x+4}+\sqrt{4}}\right|$$

Example Evaluate $\int \sin^5 x \mathrm{d}x$

Solution Use formula [33]

$$\int \sin^n x \mathrm{d}x = \frac{-1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x \mathrm{d}x$$

$$n = \_\_\_\_$$

$$\int \sin^5 x \mathrm{d}x = \frac{-1}{5}\cos x \sin^4 x + \frac{4}{5}\int \sin^3 x \mathrm{d}x$$

We need to use the formula again to find $\int \sin^3 x \mathrm{d}x$

$$\int \sin^n x \mathrm{d}x = \frac{-1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n} \int \sin^{n-2} x \mathrm{d}x$$

$$n = \_\_\_\_$$

$$\int \sin^3 x \mathrm{d}x = \frac{-1}{3} \cos x \sin^2 x + \frac{2}{3} \int \sin x \mathrm{d}x + c$$

$$= \frac{-1}{3} \cos x \sin^2 x - \frac{2}{3} \cos x + c$$

$$\int \sin^5 x = \frac{-1}{5} \cos x \sin^4 x$$

$$\frac{-4}{15} \cos x \sin^2 x - \frac{8}{15} \cos x + c$$

Example Evaluate $\int_{2e}^{3e} \frac{1}{e^2 - x^2} \mathrm{d}x$

Solution Use formula [20].

$$\int \frac{1}{a^2 - x^2} \mathrm{d}x = \frac{1}{2a} \ln \left| \frac{x+a}{x-a} \right| + c$$

$$a = \_$$

$$\int \frac{1}{e^2 - x^2} \mathrm{d}x =$$

$$\int_{2e}^{3e} \frac{1}{e^2 - x^2} \mathrm{d}x =$$

e

$$\frac{1}{2e}\ln\left|\frac{x+e}{x-e}\right|+c$$

$$\frac{1}{2e}\left[\ln\left|\frac{x+e}{x-e}\right|\right]_{2e}^{3e}$$

$$= \frac{1}{2e}\left[\ln\left(\frac{4e}{2e}\right) - \ln\left(\frac{3e}{e}\right)\right]$$
$$= \frac{1}{2e}\left[\ln 2 - \ln 3\right]$$
$$= \frac{1}{2e}\ln\left(\frac{2}{3}\right)$$

Example Evaluate $\displaystyle\int \frac{4x}{\sqrt{2x+7}}\mathrm{d}x$

Solution Use formula [16].

$$\int \frac{x}{\sqrt{ax+b}}\mathrm{d}x = \frac{2ax-4b}{3a^2}\sqrt{ax+b} + c$$

We need to re-write our integral so that we can use [16]

$$\int \frac{4x}{\sqrt{2x+7}}\mathrm{d}x = 4\int \frac{x}{\sqrt{2x+7}}\mathrm{d}x$$
$$a = \_$$
$$b = \_$$

$$\int \frac{4x}{\sqrt{2x+7}}\mathrm{d}x =$$

2

7

$$4\left[\frac{2\left(2\right)x-4\left(7\right)}{3\left(2\right)^2}\right]\sqrt{2x+7}$$

$$+\,c$$

Exercise Use the 'Table of Integrals' to evaluate the following.

$c =$ arbitrary constants

(a) $\displaystyle\int e^{2x} \sin 3x \mathrm{d}x$

$$\frac{1}{13} e^{2x} \left(2 \sin 3x - 3 \cos 3x\right) + c$$

(b) $\displaystyle\int x\sqrt{9x+1}\mathrm{d}x$

(c) $\displaystyle\int x\left(x+2\right)^4 \mathrm{d}x$

(d) $\displaystyle\int_0^1 \frac{1}{1+3e^{2x}}\mathrm{d}x$

(e) $\displaystyle\int xe^x \mathrm{d}x$

(f) $\displaystyle\int_0^{\pi/2} x \sin x \mathrm{d}x$

Exercise 3.3.4

$$\frac{2}{81}\left[\frac{(9x+1)^{5/2}}{5}-\frac{(9x+1)^{3/2}}{3}\right]+\text{c}$$

$$(x+2)^5\left(\frac{x+2}{6}-\frac{2}{5}\right)+c$$

$$\frac{1}{2}\left[2x-\ln\left(1+3e^{2x}\right)\right]+c$$

$$xe^x-e^x+c$$