

### 3.3.5 Algebraic substitution

**3.3.5.1 Indefinite integrals** Consider

$$\int 2e^{2x} dx.$$

We know that

$$\begin{aligned} \frac{d}{dx} e^{2x} &= 2e^{2x} \\ \therefore \int 2e^{2x} dx &= e^{2x} + c \end{aligned}$$

Similarly,  $\int 3 \cos 3x dx$  can be evaluated by knowing that

$$\begin{aligned} \frac{d}{dx} (\sin 3x) &= 3 \cos 3x \\ \therefore \int 3 \cos 3x dx &= \sin 3x + c \end{aligned}$$

In each of these examples we have

- a composition of functions:  $f(g(x))$
- the derivative of  $g(x)$ :  $g'(x)$

In our first example

$$\begin{aligned} f(g(x)) &= e^{2x} \text{ where } f(x) = e^x, \\ g(x) &= 2x \text{ and } g'(x) = 2. \end{aligned}$$

In our second example

$$\begin{aligned} f(g(x)) &= \cos 3x \text{ where } f(x) = \cos x, \\ g(x) &= 3x \text{ and } g'(x) = 3. \end{aligned}$$

Both these integrals have the form

$$\int f(g(x)) g'(x) dx$$

Integrals of this type can be solved by using the method of substitution by substituting  $u = g(x)$ .

How to integrate  $\int f(g(x)) \cdot g'(x) dx$

1. let  $u = g(x)$
2. Compute  $\frac{du}{dx} = g'(x)$ . Thus  
 $du = g'(x) dx$
3. Make the substitutions  
 $u = g(x)$  and  $du = g'(x) dx$  to give  
 $\int f(u) du$
4. Evaluate the integral  $\int f(u) du$
5. Replace  $u$  by  $g(x)$  so the final answer is in terms of  $x$ .

We can use this technique only when we have both  $f(g(x))$  and  $g'(x)$  present in the integrand.

We will use this method to evaluate out previous two examples.

$$\mathcal{I} = \int 2e^{2x} dx \quad \text{Let } u = \underline{\quad} = g(x)$$

$$\therefore \frac{du}{dx} = \underline{\quad} \text{ or } du = \underline{\quad}$$

$$\mathcal{I} = \int e^{2x} \cdot 2 dx$$

$$= \underline{\hspace{2cm}}$$

$$= \underline{\hspace{2cm}}$$

$$= \underline{\hspace{2cm}}$$

$$2x$$

$$2 \quad 2dx$$

$$\int e^u du$$

$$e^u + c$$

$$e^{2x} + c$$

$$\mathcal{I} = \int 3 \cos 3x dx \quad \text{Let } u = \underline{\quad} = g(x)$$

$$\therefore \frac{du}{dx} = \underline{\quad} \text{ or } du = \underline{\quad}$$

$$\mathcal{I} = \int \cos 3x \cdot 3 dx$$

$$= \underline{\hspace{2cm}}$$

$$= \underline{\hspace{2cm}}$$

$$= \sin 3x + c$$

**Example** Evaluate the following integrals using an appropriate substitution.

$$1. \int 2x (x^2 + 1)^{5/2} dx$$

$$2. \int -\sin x \cdot \cos^3 x dx$$

 $3x$ 
 $3$ 
 $3dx$ 

$$\int \cos u du$$

$$\sin u + c$$

$$\mathcal{I} = \int 2x (x^2 + 1)^{5/2} dx$$

$$u = x^2 + 1$$

$$du = 2x dx$$

$$\begin{aligned}\mathcal{I} &= \int u^{5/2} du \\ &= \frac{2}{7} u^{7/2} + c \\ &= \frac{2}{7} (x^2 + 1)^{7/2} + c.\end{aligned}$$

$c$  is an arbitrary constant.

$$\mathcal{I} = \int -\sin x \cdot \cos^3 x dx$$

$$u = \cos x$$

$$du = -\sin x dx$$

$$\begin{aligned}\mathcal{I} &= \int u^3 du = \frac{u^4}{4} + c \\ &= \frac{\cos^4 x}{4} + c\end{aligned}$$

$c$  is an arbitrary constant.

If  $g'(x)$  differs from what we have in the integrand by a *constant factor only* our method will still work.

$$\mathcal{I} = \int x\sqrt{1+x^2}dx \quad \text{Let } u = x^2 + 1$$

$$\therefore du = 2xdx$$

$$\begin{aligned} \mathcal{I} &= \frac{1}{2} \int 2x\sqrt{1+x^2}dx \\ &= \frac{1}{2} \int \sqrt{u}du \\ &= \frac{1}{2} \int u^{1/2}du \\ &= \frac{1}{2} \left( \frac{2}{3}u^{3/2} \right) + c \\ &= \frac{1}{3}u^{3/2} + c \\ &= \frac{1}{3}(1+x^2)^{3/2} + c \end{aligned}$$

### Example

$$\mathcal{I} = \int \frac{\cos \sqrt{x}}{\sqrt{x}} dx$$

$$\text{Let } u = \underline{\hspace{2cm}}$$

$$\therefore du = \underline{\hspace{2cm}}$$

$$\begin{aligned} & x^{1/2} \\ & \frac{1}{2}x^{-1/2}dx \\ \Rightarrow dx &= \frac{2du}{x^{-1/2}} = 2x^{1/2}du \end{aligned}$$

$$\begin{aligned} \mathcal{I} &= \int \frac{\cos u \cdot 2x^{1/2}}{x^{1/2}} du \\ &= 2 \int \cos(u) du \\ &= 2 \sin(u) + c \\ &= 2 \sin(\sqrt{x}) + c \end{aligned}$$

$c$  an arbitrary constant.

### 3.3.5.2 Definite Integrals

$$\begin{aligned}\mathcal{I} &= \int_a^b f(g(x)) \cdot g'(x) dx \\ &= \int_{x=a}^{x=b} f(g(x)) \cdot g'(x) dx \\ &= \int_{g(a)}^{g(b)} f(u) du\end{aligned}$$

When we make the substitution  $u = g(x)$  in a definite integral, the x-limits of integration are affected. We can deal with this in one of two ways

1. Change the limits of integration, which eliminates the need to back-substitute.
2. Evaluate the indefinite integral first and then substitute in the x-limits.

**Example** Evaluate  $\int_0^2 x(x^2 + 1)^3 dx$ .

(Method One)

$$\mathcal{I} = \int_{x=0}^{x=2} x(x^2 + 1)^3 dx$$

$$\text{Let } u = x^2 + 1 \quad \therefore du = 2x dx$$

$$\mathcal{I} = \frac{1}{2} \int_{x=0}^{x=2} 2x(x^2 + 1)^3 dx$$

$$= \frac{1}{2} \int_{x=0}^{x=2} u^3 du$$

$$\text{Now } x = 0 \Rightarrow u = \underline{1}$$

$$\text{Now } x = 2 \Rightarrow u = \underline{5}$$

$$\begin{aligned}\mathcal{I} &= \frac{1}{2} \int_{u=1}^{u=5} u^3 du \\ &= \frac{1}{2} \left[ \frac{u^4}{4} \right]_{u=1}^{u=5} = \frac{1}{8} [5^4 - 1^4] \\ &= \frac{224}{8}\end{aligned}$$

$$\int_0^2 x (x^2 + 1)^3 dx. \text{ (Method Two)}$$

(1) Determine the indefinite integral

$$\mathcal{I} = \int x (x^2 + 1)^3 dx$$

Let  $u = x^2 + 1$

$$\begin{aligned} \mathcal{I} &= \frac{1}{2} \int u^3 du = \underline{\hspace{2cm}} \\ &= \underline{\hspace{2cm}} \end{aligned}$$

(2) Determine the definite integral

$$\begin{aligned} \int_0^2 x (x^2 + 1)^3 &= \frac{1}{8} \left[ \underline{(x^2 + 1)^4} \right]_0^2 \\ &= \frac{1}{8} (\underline{\hspace{2cm}}) \end{aligned}$$

$$\frac{1}{2} \cdot \frac{u^4}{4} + c$$

$$\frac{1}{8} (x^2 + 1)^4 + c$$

$$5^4 - 1^4$$

### 3.3.6 Integrals of the form

$$\int \frac{f'(x)}{f(x)} dx$$

Consider the following derivatives

$$\begin{aligned} \frac{d}{dx} [\ln |x^3 + 1|] &= \frac{1}{x^3 + 1} \cdot \frac{d}{dx} (x^3 + 1) \\ &= \frac{3x^2}{x^3 + 1} \end{aligned}$$

$$\begin{aligned} \frac{d}{dx} [\ln |\sin x|] &= \frac{1}{\sin x} \cdot \frac{d}{dx} (\sin x) \\ &= \frac{\cos x}{\sin x} \end{aligned}$$

$$\begin{aligned} \frac{d}{dx} [\ln |x^2 + 2x|] &= \frac{1}{x^2 + 2x} \cdot \frac{d}{dx} (x^2 + 2x) \\ &= \frac{2x + 2}{x^2 + 2x} \end{aligned}$$

In each of these derivatives, involving the ln function, the answer is a quotient in which the numerator is the derivative of the denominator.

**True or False?**

$$\int \frac{f'(x)}{f(x)} dx = \ln |f(x)|$$

**Examples** Evaluate the following integrals

1.  $\int \frac{x}{x^2 + 4} dx$

$$\frac{1}{2} \ln |x^2 + 4|$$

2.  $\int_0^{\pi/2} \frac{\cos x}{2 + \sin x} dx$

$$\ln 1.5$$

3.  $\int \tan x dx$

$$\ln |\sec x| + c$$

4.  $\int_0^{\ln 2} \frac{e^x}{1 + e^x} dx$

$$\ln 1.5$$

1. Evaluate the integral  $\int \frac{x}{x^2 + 4} dx$ .

**Solution.**

$$\mathcal{I} = \int \frac{x}{x^2 + 4} dx,$$

$$\text{Let } u = x^2 + 4,$$

$$\text{Then } \frac{du}{dx} = 2x \Rightarrow dx = \frac{du}{2x},$$

$$\begin{aligned} \text{Thus } \mathcal{I} &= \int \frac{x}{u} \cdot \frac{du}{2x}, \\ &= \frac{1}{2} \int \frac{1}{u} du, \\ &= \frac{1}{2} \ln |u| + c, \\ &= \frac{1}{2} \ln |x^2 + 4| + c, \end{aligned}$$

where  $c$  is an arbitrary constant.

2. Evaluate the integral

$$\int_0^{\pi/2} \frac{\cos x}{2 + \sin x} dx.$$

**Solution.**

$$\mathcal{I} = \int_0^{\pi/2} \frac{\cos x}{2 + \sin x} dx,$$

$$\text{Let } u = 2 + \sin x,$$

$$\text{Then } \frac{du}{dx} = \cos x \Rightarrow dx = \frac{du}{\cos x},$$

$$x = 0 \Rightarrow u = 2 + \sin(0) = 2,$$

$$x = \frac{\pi}{2} \Rightarrow u = 2 + \sin\left(\frac{\pi}{2}\right) = 3,$$

$$\text{Thus } \mathcal{I} = \int_2^3 \frac{\cos x}{u} \cdot \frac{du}{\cos x},$$

$$= \int_2^3 \frac{1}{u} du,$$

$$= [\ln |u|]_2^3,$$

$$= \ln 3 - \ln 2 = \ln 1.5.$$

3. Evaluate the integral  $\int \tan x dx$ .

**Solution.**

$$\mathcal{I} = \int \tan x dx = \int \frac{\sin x}{\cos x} dx,$$

Let  $u = \cos x$ ,

$$\text{Then } \frac{du}{dx} = -\sin x \Rightarrow dx = -\frac{du}{\sin x},$$

$$\text{Thus } \mathcal{I} = \int \frac{\sin x}{u} \cdot -\frac{du}{\sin x},$$

$$= -\int \frac{1}{u} du,$$

$$= -\ln |u| + c,$$

$$= -\ln |\cos x| + c,$$

$$= \ln |\sec x| + c,$$

where  $c$  is an arbitrary constant.

4. Evaluate the integral

$$\int_0^{\ln 2} \frac{e^x}{1+e^x} dx.$$

**Solution.**

$$\mathcal{I} = \int_0^{\ln 2} \frac{e^x}{1+e^x} dx,$$

$$\text{Let } u = 1 + e^x,$$

$$\text{Then } \frac{du}{dx} = e^x \Rightarrow dx = \frac{du}{e^x},$$

$$x = 0 \Rightarrow u = 1 + e^0 = 2,$$

$$x = \ln 2 \Rightarrow u = 1 + e^{\ln 2} = 3,$$

$$\text{Thus } \mathcal{I} = \int_2^3 \frac{e^x}{u} \cdot \frac{du}{e^x},$$

$$= \int_2^3 \frac{1}{u} du,$$

$$= [\ln |u|]_2^3,$$

$$= \ln 3 - \ln 2 = \ln 1.5.$$

### **3.3.7 Exercises**

The following questions are about the key ideas in this section.