3.3.5 Algebraic substitution

3.3.5.1 Indefinite integrals Consider $\int 2e^{2x}\mathrm{d}x$.

We know that

$$\frac{\mathrm{d}}{\mathrm{d}x}e^{2x} = 2e^{2x}$$

$$\therefore \quad \int 2e^{2x}\mathrm{d}x = e^{2x} + c$$

Similarly, $\int 3\cos 3x\,\mathrm{d}x$ can be evaluated by knowing that

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin 3x) = 3\cos 3x$$

$$\therefore \quad \int 3\cos 3x\mathrm{d}x = \sin 3x + c$$

In each of these examples we have

- a composition of functions: $f(g(x))$
- the derivative of $g(x)$: $g'(x)$

In our first example
$f(g(x)) = e^{2x}$ where $f(x) = e^x$,
$g(x) = 2x$ and $g'(x) = 2$.

In our second example
$f(g(x)) = \cos 3x$ where $f(x) = \cos x$,
$g(x) = 3x$ and $g'(x) = 3$.

Both these integrals have the form

$$\int f\left(g\left(x\right)\right) g'\left(x\right) \mathrm{d}x$$

Integrals of this type can be solved by using the method of <u>substitution</u> by substituting $\underline{u = g(x)}$.

How to integrate $\int f(g(x)) \cdot g'(x)\mathrm{d}x$

1. let $u = g(x)$
2. Compute $\dfrac{\mathrm{d}u}{\mathrm{d}x} = g'(x)$. Thus $\mathrm{d}u = g'(x)\mathrm{d}x$
3. Make the substitutions $u = g(x)$ and $\mathrm{d}u = g'(x)\mathrm{d}x$ to give $\int f(u)\mathrm{d}u$
4. Evaluate the integral $\int f(u)\mathrm{d}u$
5. Replace u by $g(x)$ so the final answer is in terms of x.

We can use this technique <u>only</u> when we have <u>both</u> $f(g(x))$ <u>and</u> $g'(x)$ present in the integrand.

We will use this method to evaluate out previous two examples.

$$\mathcal{I} = \int 2e^{2x}\mathrm{d}x \qquad \text{Let} \quad u = \_\_ = g(x)$$

$$\therefore \frac{\mathrm{d}u}{\mathrm{d}x} = \_ \text{ or } \mathrm{d}u = \_\_\_$$

$$\mathcal{I} = \int e^{2x} \cdot 2\mathrm{d}x$$

$$= \frac{}{\_\_\_\_\_\_}$$

$$= \_\_\_\_\_$$

$$= \_\_\_\_\_\_$$

$2x$

2 $2\mathrm{d}x$

$$\int e^u \mathrm{d}u$$

$$e^u + c$$

$$e^{2x} + c$$

$$\mathcal{I} = \int 3\cos 3x \mathrm{d}x \qquad \text{Let} \quad u = \underline{\qquad} = g(x)$$

$$\therefore \frac{\mathrm{d}u}{\mathrm{d}x} = \underline{\ } \text{ or } \mathrm{d}u = \underline{\qquad}$$

$$\mathcal{I} = \int \cos 3x \cdot 3\mathrm{d}x$$

$$= \underline{\qquad\qquad}$$

$$= \underline{\qquad\qquad}$$

$$= \sin 3x + c$$

Example Evaluate the following integrals using an appropriate substitution.

1. $\int 2x\left(x^2+1\right)^{5/2} \mathrm{d}x$
2. $\int -\sin x \cdot \cos^3 x \mathrm{d}x$

$3x$

3 $3\mathrm{d}x$

$$\int \cos u \mathrm{d}u$$

$\sin u + c$

$$\mathcal{I} = \int 2x\left(x^2+1\right)^{5/2}\mathrm{d}x$$

$$
\begin{aligned}
u &= x^2 + 1 \\
\mathrm{d}u &= 2x\mathrm{d}x \\
\mathcal{I} &= \int u^{5/2}\mathrm{d}u \\
&= \frac{2}{7}u^{7/2} + c \\
&= \frac{2}{7}\left(x^2 + 1\right)^{7/2} + c.
\end{aligned}
$$

c is an arbitrary constant.

$$\mathcal{I} = \int -\sin x \cdot \cos^3 x \mathrm{d}x$$

$$
\begin{aligned}
u &= \cos x \\
\mathrm{d}u &= -\sin x \mathrm{d}x \\
\mathcal{I} &= \int u^3 \mathrm{d}u = \frac{u^4}{4} + c \\
&= \frac{\cos^4 x}{4} + c
\end{aligned}
$$

c is an arbitrary constant.

If $g'(x)$ differs from what we have in the integrand by a *constant factor only* our method will still work.

$$\mathcal{I} = \int x\sqrt{1+x^2}\mathrm{d}x \qquad \text{Let} \quad u = x^2 + 1$$

$$\therefore \mathrm{d}u = 2x\mathrm{d}x$$

$$\begin{aligned}
\mathcal{I} &= \frac{1}{2}\int 2x\sqrt{1+x^2}\mathrm{d}x \\
&= \frac{1}{2}\int \sqrt{u}\mathrm{d}u \\
&= \frac{1}{2}\int u^{1/2}\mathrm{d}u \\
&= \frac{1}{2}\left(\frac{2}{3}u^{3/2}\right) + c \\
&= \frac{1}{3}u^{3/2} + c \\
&= \frac{1}{3}\left(1+x^2\right)^{3/2} + c
\end{aligned}$$

Example

$$\mathcal{I} = \int \frac{\cos\sqrt{x}}{\sqrt{x}}\mathrm{d}x$$

$$\text{Let} \quad u = \underline{\qquad}$$

$$\therefore \mathrm{d}u = \frac{}{\rule{4cm}{0.4pt}}$$

$$x^{1/2}$$

$$\frac{1}{2}x^{-1/2}\mathrm{d}x$$

$$\Rightarrow \mathrm{d}x = \frac{2\mathrm{d}u}{x^{-1/2}} = 2x^{1/2}\mathrm{d}u$$

$$
\begin{aligned}
\mathcal{I} &= \int \frac{\cos u \cdot 2x^{1/2}}{x^{1/2}} \, \mathrm{d}u \\
&= 2 \int \cos(u) \, \mathrm{d}u \\
&= 2 \sin(u) + c \\
&= 2 \sin\left(\sqrt{x}\right) + c
\end{aligned}
$$

c an arbitrary constant.

3.3.5.2 Definite Integrals

$$\begin{aligned}\mathcal{I} &= \int_a^b f\left(g\left(x\right)\right) \cdot g'\left(x\right) \mathrm{d}x \\ &= \int_{x=a}^{x=b} f\left(g\left(x\right)\right) \cdot g'\left(x\right) \mathrm{d}x \\ &= \int_{g(a)}^{g(b)} f\left(u\right) \mathrm{d}u\end{aligned}$$

When we make the substitution $u = g(x)$ in a definite integral, the x-limits of integration are affected. We can deal with this in one of two ways

1. Change the limits of integration, which eliminates the need to back-substitute.
2. Evaluate the indefinite integral first and then substitute in the x-limits.

Example Evaluate $\displaystyle\int_0^2 x\left(x^2+1\right)^3 \mathrm{d}x.$

(Method One)

$$\mathcal{I} = \int_{x=0}^{x=2} x\left(x^2+1\right)^3 \mathrm{d}x$$

$$\text{Let } u = x^2+1 \qquad \therefore\ \mathrm{d}u = 2x\mathrm{d}x$$

$$\mathcal{I} = \frac{1}{2}\int_{x=0}^{x=2} 2x\left(x^2+1\right)^3 \mathrm{d}x$$

$$= \frac{1}{2}\int_{x=0}^{x=2} u^3 \mathrm{d}u$$

$$\text{Now} \quad x = 0 \Rightarrow u = \underline{1}$$

$$\text{Now} \quad x = 2 \Rightarrow u = \underline{5}$$

$$\mathcal{I} = \frac{1}{2}\int_{u=\underline{1}}^{u=\underline{5}} u^3 \mathrm{d}u$$

$$= \frac{1}{2}\left[\frac{u^4}{4}\right]_{u=\underline{1}}^{u=\underline{5}} = \frac{1}{8}\left[\underline{5^4 - 1^4}\right]$$

$$= \underline{\frac{224}{8}}$$

$\int_0^2 x\left(x^2+1\right)^3 \mathrm{d}x.$ (Method Two)

(1) Determine the indefinite integral

$$\mathcal{I} = \int x\left(x^2+1\right)^3 \mathrm{d}x$$

Let $u = x^2 + 1$

$$\mathcal{I} = \frac{1}{2}\int u^3 \mathrm{d}u = \underline{\hspace{3cm}}$$

$$= \underline{\hspace{4cm}}$$

(2) Determine the definite integral

$$\int_0^2 x\left(x^2+1\right)^3 = \underline{\frac{1}{8}\left[\left(x^2+1\right)^4\right]_0^2}$$

$$= \frac{1}{8}\left(\underline{\hspace{2cm}}\right)$$

$$\frac{1}{2} \cdot \frac{u^4}{4} + c$$

$$\frac{1}{8}\left(x^2 + 1\right)^4 + c$$

$$5^4 - 1^4$$

3.3.6 Integrals of the form $\displaystyle\int \frac{f'(x)}{f(x)}\mathbf{d}x$

Consider the following derivatives

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\ln\left|x^3+1\right|\right] = \frac{1}{x^3+1}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(x^3+1\right)$$
$$= \frac{3x^2}{x^3+1}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\ln\left|\sin x\right|\right] = \frac{1}{\sin x}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin x\right)$$
$$= \frac{\cos x}{\sin x}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\ln\left|x^2+2x\right|\right] = \frac{1}{x^2+2x}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2+2x\right)$$
$$= \frac{2x+2}{x^2+2x}$$

In each of these derivatives, involving the ln function, the answer is a quotient in which the numerator is the derivative of the denominator.

True or **False**?

$$\int \frac{f'(x)}{f(x)} \mathrm{d}x = \ln|f(x)|$$

Examples Evaluate the following integrals

1. $\displaystyle\int \frac{x}{x^2+4} \mathrm{d}x$

2. $\displaystyle\int_0^{\pi/2} \frac{\cos x}{2+\sin x} \mathrm{d}x$

3. $\displaystyle\int \tan x \mathrm{d}x$

4. $\displaystyle\int_0^{\ln 2} \frac{e^x}{1+e^x} \mathrm{d}x$

$$\frac{1}{2}\ln\left|x^2+4\right|$$

$$\ln 1.5$$

$$\ln|\sec x|+c$$

$$\ln 1.5$$

1. Evaluate the integral $\int \frac{x}{x^2+4} \mathrm{d}x$.

Solution.

$$
\begin{aligned}
\mathcal{I} &= \int \frac{x}{x^2+4}\mathrm{d}x, \\
\text{Let}\, u &= x^2+4, \\
\text{Then}\, \frac{\mathrm{d}u}{\mathrm{d}x} &= 2x \Rightarrow \mathrm{d}x = \frac{\mathrm{d}u}{2x}, \\
\text{Thus}\, \mathcal{I} &= \int \frac{x}{u}\cdot\frac{\mathrm{d}u}{2x}, \\
&= \frac{1}{2}\int \frac{1}{u}\mathrm{d}u, \\
&= \frac{1}{2}\ln|u| + c, \\
&= \frac{1}{2}\ln\left|x^2+4\right| + c,
\end{aligned}
$$

where c is an arbitrary constant.

2. Evaluate the integral

$$\int_0^{\pi/2} \frac{\cos x}{2 + \sin x} \mathrm{d}x.$$

Solution.

$$
\begin{aligned}
\mathcal{I} &= \int_0^{\pi/2} \frac{\cos x}{2+\sin x}\mathrm{d}x,\\
\text{Let } u &= 2+\sin x,\\
\text{Then } \frac{\mathrm{d}u}{\mathrm{d}x} &= \cos x \Rightarrow \mathrm{d}x = \frac{\mathrm{d}u}{\cos x},\\
x &= 0 \Rightarrow u = 2+\sin(0) = 2,\\
x &= \frac{\pi}{2} \Rightarrow u = 2+\sin\left(\frac{\pi}{2}\right) = 3,\\
\text{Thus } \mathcal{I} &= \int_2^3 \frac{\cos x}{u}\cdot\frac{\mathrm{d}u}{\cos x},\\
&= \int_2^3 \frac{1}{u}\mathrm{d}u,\\
&= \left[\ln|u|\right]_2^3,\\
&= \ln 3 - \ln 2 = \ln 1.5.
\end{aligned}
$$

3. Evaluate the integral $\displaystyle\int \tan x \, \mathrm{d}x$.

Solution.

$$\begin{aligned}
\mathcal{I} &= \int \tan x \mathrm{d}x = \int \frac{\sin x}{\cos x}\mathrm{d}x, \\
\text{Let } u &= \cos x, \\
\text{Then } \frac{\mathrm{d}u}{\mathrm{d}x} &= -\sin x \Rightarrow \mathrm{d}x = -\frac{\mathrm{d}u}{\sin x}, \\
\text{Thus } \mathcal{I} &= \int \frac{\sin x}{u} \cdot -\frac{\mathrm{d}u}{\sin x}, \\
&= -\int \frac{1}{u}\mathrm{d}u, \\
&= -\ln|u| + c, \\
&= -\ln|\cos x| + c, \\
&= \ln|\sec x| + c,
\end{aligned}$$

where c is an arbitrary constant.

4. Evaluate the integral

$$\int_0^{\ln 2} \frac{e^x}{1+e^x}\mathrm{d}x.$$

Solution.

$$\mathcal{I} = \int_0^{\ln 2} \frac{e^x}{1+e^x} \mathrm{d}x,$$

$$\text{Let } u = 1 + e^x,$$

$$\text{Then } \frac{\mathrm{d}u}{\mathrm{d}x} = e^x \Rightarrow \mathrm{d}x = \frac{\mathrm{d}u}{e^x},$$

$$x = 0 \Rightarrow u = 1 + e^0 = 2,$$

$$x = \ln 2 \Rightarrow u = 1 + e^{\ln 2} = 3,$$

$$\text{Thus } \mathcal{I} = \int_2^3 \frac{e^x}{u} \cdot \frac{\mathrm{d}u}{e^x},$$

$$= \int_2^3 \frac{1}{u} \mathrm{d}u,$$

$$= \left[\ln |u|\right]_2^3,$$

$$= \ln 3 - \ln 2 = \ln 1.5.$$

3.3.7 Exercises

The following questions are about the key ideas in this section.