

Fundamentals Lecture Eight

Basic Trig Identities :

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Basic Trigonometric Identities

(i) Pythagorean Formulae

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$1 + \cot^2 \theta = \operatorname{cosec}^2 \theta$$

$$\tan^2 \theta + 1 = \sec^2 \theta$$

Question: How are the second and third of these formulae derived from the first?

Example

Show that

$$(\sin \theta + \cos \theta)^2 + (\sin \theta - \cos \theta)^2 = 2$$

Solution

$$\text{LHS} = (\sin \theta + \cos \theta)^2 + (\sin \theta - \cos \theta)^2$$

$$=$$
$$=$$
$$=$$
$$=$$
$$= \text{RHS} \quad \square$$

$$\begin{aligned} & + \sin^2 \theta + \cos^2 \theta + 2 \cos \theta \sin \theta \\ & + \sin^2 \theta + \cos^2 \theta - 2 \cos \theta \sin \theta \\ & 2 \sin^2 \theta + 2 \cos^2 \theta \\ & 2 (\sin^2 \theta + \cos^2 \theta) \\ & 2 \end{aligned}$$

Example

Verify that

$$\frac{1 - \sin \theta}{\cos \theta} = \frac{\cos \theta}{1 + \sin \theta} \quad \text{Ex1N Q29(c)}$$

Solution Multiply both sides of the equation by $\cos \theta (1 + \sin \theta)$ to obtain

$$(1 - \sin \theta) (1 + \sin \theta) = \cos \theta \cos \theta$$

$$1 - \sin^2 \theta = \cos^2 \theta$$

$$1 = \cos^2 \theta + \sin^2 \theta$$

$$1 = 1$$

because $\sin^2 \theta + \cos^2 \theta = 1$ by the
Pythagorean formula

(ii) Sum and Difference of Angle Formulae

$$\sin (A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos (A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan (A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

Example Find an exact value for

$$\cos \frac{7\pi}{12}$$

Solution Now, $\frac{7\pi}{12} = \frac{\pi}{3} + \frac{\pi}{4}$

$$\therefore \cos \frac{7\pi}{12} = \cos \left(\frac{\pi}{3} + \frac{\pi}{4} \right)$$

$$=$$

$$=$$

$$= \frac{1 - \sqrt{3}}{2\sqrt{2}}$$

$$\begin{aligned} & \cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right) \\ & \frac{1}{2} \cdot \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} \end{aligned}$$

Example Find an exact value for $\sin \frac{\pi}{12}$

Solution Now, $\frac{7\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$

$$\begin{aligned}\therefore \sin \frac{\pi}{12} &= \sin \left(\frac{\pi}{3} - \frac{\pi}{4} \right) \\ &= \\ &= \\ &= \\ &= \frac{\sqrt{2}}{4} (\sqrt{3} - 1)\end{aligned}$$

$$\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)$$

$$\frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} - \frac{1}{2} \cdot \frac{1}{\sqrt{2}}$$

$$\frac{1}{2\sqrt{2}} (\sqrt{3} - 1)$$

(iii) Double Angle Formulae

$$\begin{aligned}\sin 2\theta &= 2 \sin \theta \cos \theta \\ \cos 2\theta &= \cos^2 \theta - \sin^2 \theta \\ &= 2 \cos^2 \theta - 1 \\ &= 1 - 2 \sin^2 \theta \\ \tan 2\theta &= \frac{2 \tan \theta}{1 - \tan^2 \theta}\end{aligned}$$

These formula can be obtained by letting $A = B = \theta$ in (ii)

Example Prove that $\frac{1 - \cos 2\theta}{\sin 2\theta} = \tan \theta$.

Solution

$$\text{LHS} = \frac{1 - \cos 2\theta}{\sin 2\theta}$$

Now $\cos 2\theta = 1 - 2\sin^2 \theta$ (Double Angle Formulae)

$$\therefore \text{LHS} =$$

$$=$$

$$=$$

because $\sin 2\theta = 2\sin \theta \cos \theta$ (Double Angle Formulae)

$$=$$

$$= \text{RHS} \quad \square$$

$$\frac{1 - (1 - 2 \sin^2 \theta)}{\sin 2\theta}$$
$$\frac{2 \sin^2 \theta}{\sin 2\theta}$$
$$\frac{2 \sin^2 \theta}{2 \sin \theta \cos \theta}$$

$$\frac{\sin \theta}{\cos \theta}$$

Half-angle Formulae

$$\begin{aligned}\sin^2 \frac{\theta}{2} &= \frac{1 - \cos \theta}{2} \\ \cos^2 \frac{\theta}{2} &= \frac{1 + \cos \theta}{2} \\ \tan \frac{\theta}{2} &= \frac{1 - \cos \theta}{\sin \theta} \\ &= \frac{\sin \theta}{1 + \cos \theta}\end{aligned}$$

Note: From the half-angle formulae

$$\begin{aligned}\sin \frac{\theta}{2} &= \pm \sqrt{\frac{1 - \cos \theta}{2}} \\ \cos \frac{\theta}{2} &= \pm \sqrt{\frac{1 + \cos \theta}{2}}\end{aligned}$$

The ‘+’ or ‘-’ sign is chosen to be compatible with the known quadrant of $\frac{\theta}{2}$.

Example

1. Explain why $\cos \frac{5\pi}{3} = \frac{1}{2}$
2. Hence find an exact value for $\cos \frac{5\pi}{6}$ using the half-angle formulae.

Solution We need to use the fact that $\frac{5\pi}{6} = \frac{1}{2} \cdot \left(\frac{5\pi}{3}\right)$ and that

$$\cos^2 \left(\frac{\theta}{2} \right) = \frac{1 + \cos \theta}{2}$$
$$\Rightarrow \cos \frac{\theta}{2} = \pm \sqrt{\frac{1 + \cos \theta}{2}}$$

$$\cos \frac{\theta}{2} = \pm \sqrt{\frac{1 + \cos \theta}{2}}$$

Note that $\frac{5\pi}{6}$ is in the 2nd quadrant so that $\cos \frac{5\pi}{6}$ is negative. Let $\theta = \frac{5\pi}{3}$

$$\therefore \cos \frac{5\pi}{6} =$$

$$=$$
$$=$$
$$=$$

$$- \left[\frac{1 + \cos\left(\frac{5\pi}{3}\right)}{2} \right]^{1/2}$$

$$- \left[\frac{1}{2} + \frac{1}{4} \right]^{1/2}$$

$$- \left[\frac{3}{4} \right]^{1/2}$$

$$- \frac{\sqrt{3}}{2}$$

Products as Sums or Differences

$$2 \cos A \cos B = \cos(A+B) + \cos(A-B)$$

$$2 \sin A \sin B = \cos(A-B) - \cos(A+B)$$

$$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$$

$$2 \cos A \sin B = \sin(A+B) - \sin(A-B)$$

These formulae are obtained by the addition or subtraction of the formulae in (ii)

Examples

Write the following products as a sum or difference.

a $2 \cos 7x \cos 5x$

b $\cos 4x \sin x$

Solutions

a Let $A = 7x$ and $B = 5x$. Then as

$$2 \cos A \cos B = \cos(A+B) + \cos(A-B)$$

we have

$$\begin{aligned} 2 \cos 7x \cos 5x &= \cos(7x+5x) + \cos(7x-5x) \\ &= \cos(12x) + \cos(2x) \end{aligned}$$

b Let $A = 4x$ and $B = x$. Then as

$$\cos A \sin B = \frac{1}{2} [\sin(A+B) - \sin(A-B)]$$

we have

$$\begin{aligned} \cos 4x \sin x &= \frac{1}{2} [\sin(4x+x) - \sin(4x-x)] \\ &= \frac{1}{2} [\sin(5x) - \sin(3x)] \end{aligned}$$

Half-angle tangent formula

$$\begin{aligned} \text{Let } t &= \tan \frac{x}{2}, \\ \text{then } \sin x &= \frac{2t}{1+t^2} \\ \cos x &= \frac{1-t^2}{1+t^2} \\ \tan x &= \frac{2t}{1-t^2} \end{aligned}$$

This substitution is useful in integrating rational functions of sine and cosine (MATH 142).

Example Find an expression for $\frac{1}{1 + \cos x}$ by letting $t = \tan \frac{x}{2}$.

Solution Now, $\cos x = \frac{1-t^2}{1+t^2}$. Hence

$$\begin{aligned} \frac{1}{1 + \cos x} &= \\ &= \\ &= \end{aligned}$$

$$\frac{1}{1 + \frac{1-t^2}{1+t^2}}$$
$$\frac{1}{\frac{1+t^2}{1+t^2} + \frac{1-t^2}{1+t^2}}$$
$$\frac{1+t^2}{2}.$$

Exercises on Trig Identities

Exercise 1.13.7
pages 44 & 45.

For each of the questions do as many of the subquestions as you require in order to gain mastery of the basic technique.

You *need* to do these exercises!