

# Fundamentals

## Lecture Seven

**Remainder Theorem** : polynomial division. 1.4.9 (page 1-14)

- Determine the remainder when the polynomial  $x^3 + 5x^2 - 2x - 24$  is divided by the polynomial  $x - 1$ .

**Factor Theorem** : Finding factors of polynomials. 1.4.9 (page 1-14)

- Show that  $(x - 3)$  is a factor of  $x^3 - 5x^2 - 2x + 24$  and hence determine the remaining factors.

Consider out previous divisions

$$\text{i } (x^2 + 5x - 36) \div (x - 4) = x + 9$$

$$\text{ii } \frac{x^3 - 2x^2 - x + 2}{x + 1} = x^2 - 3x + 2$$

$$\text{iii } (x^3 + 1) \div (x + 1) = x^2 - x + 1$$

$$\text{iv } \frac{3x^3 - 7x^2 + 2x + 4}{x - 3} = 3x^2 + 2x + 8 + \frac{28}{x - 3}$$

Note that in each case when our given polynomial is divided by  $x - a$  the quotient (or answer) is a polynomial of *one degree less* than the original together with a remainder (which was zero in (i), (ii) and (iii) and 28 in (iv)).

We can write this as an equation:

$$\frac{f(x)}{x - a} = q(x) + \frac{r}{x - a} \quad \text{OR}$$

$$f(x) = (x - a)q(x) + r$$

where  $f(x)$  is our given polynomial,  $q(x)$  is the polynomial produced by the division and  $r$  is the remainder.

If we let  $x = a$  in

$$f(x) = (x - a)q(x) + r \quad \text{we get}$$

$$\begin{aligned} f(a) &= (a - a)q(a) + r \\ &= r \quad \text{the remainder} \end{aligned}$$

So, to find the remainder resulting from a division we don't have to perform the division we can simply use the rule

$$f(a) = r$$

For example, we know that

$(3x^3 - 7x^2 + 2x + 4) \div (x - 3)$  results in a remainder of 28. We will check if we get the same remainder using this new rule

Now,  $f(x) = 3x^3 - 7x^2 + 2x + 4$

and  $x - 3 = x - a \Rightarrow a = 3$

$$f(3) =$$

$$=$$

$$= 28$$

**Example** Determine the remainder in each of the following cases.

$$1. (x^3 + 5x^2 - 2x - 24) \div (x - 1).$$

$$a = \underline{\hspace{2cm}}$$

$$\therefore r = f(\quad)$$

$$=$$

$$=$$

$$=$$

$$2. (x^3 + 5x^2 - 2x - 24) \div (x + 1)$$

$$a = \underline{\hspace{2cm}}$$

$$\therefore r = f(\quad)$$

$$=$$

$$=$$

$$=$$

–20

–18

$$3. (x^3 + 5x^2 - 2x - 24) \div (x - 2)$$

$$a = \underline{\quad}$$

$$\therefore r = f(\quad)$$

$$=$$

$$=$$

$$=$$

Key point 1: Let  $\mathcal{P}(x)$  be a polynomial.

Then the remainder when  $\mathcal{P}(x)/x - a$  is  $\mathcal{P}(a)$ .

Key point 2: If  $\mathcal{P}(a) = 0$  then  $x - a$  is a *factor* of  $\mathcal{P}(x)$ .

0

**Factor Theorem** Why is being able to find the remainder easily important? In our example

$$1224 \div 8 = 153$$

the *remainder is zero* and so

$$1224 = 8 \times 153$$

i.e. 8 and 153 are *factors* of 1224.

Similarly, (i), (ii) and (iii) have *remainders of zero*. So,

i  $x^2 + 5x - 36$  has *factors*  $(x - 4)$  and  $(x + 9)$

ii  $x^3 - 2x^2 - x + 2$  has *factors*  $(x + 1)$  and  $(x^2 - 3x + 2)$

iii  $x^3 + 1$  has *factors*  $(x + 1)$  and  $(x^2 - x + 2)$

Therefore, if  $f(a) = 0$ , then  $(x - a)$  is a factor of  $f(x)$  For example, if

$$f(x) = x^3 + 2x^2 - 14x + 12$$

$$\begin{aligned} \text{then } f(2) &= 2^3 + 2 \cdot 2^2 - 14 \cdot 2 + 12 \\ &= 0. \end{aligned}$$

Therefore,  $(x - 2)$  is a factor of  $f(x)$ . The remaining factor can be found by long division.

$$x - 2 \overline{) x^3 + 2x^2 - 14x + 12}$$

$$\begin{aligned} \therefore f(x) &= x^3 + 2x^2 - 14x + 12 \\ &= (x - 2) \underline{\hspace{2cm}} \end{aligned}$$

$$(x^2 + 4x - 6)$$

**Example** Show that  $(x - 3)$  is a factor of  $f(x) = x^3 - 5x^2 - 2x + 24$  and hence determine the remaining factor(s).

**Solution**  $a = \underline{\hspace{2cm}}$

$$\begin{aligned} \therefore r &= f(\quad) \\ &= \\ &= \end{aligned}$$

$\therefore (x - 3)$  is a factor of  $f(x)$ . To determine further factor(s) use long division

$$x - 3 \left| \overline{x^3 - 5x^2 - 2x + 24} \right.$$

$$\begin{aligned}
 f(x) &= x^3 - 5x^2 - 2x + 24 \\
 &= (x - 3)(x^2 - 2x - 8) \\
 &=
 \end{aligned}$$

What happens if we are not given the first factor? Let's look at an example

**Example** To attempt to find the linear factors of

$$f(x) = x^3 + 5x^2 - 2x - 24$$

we look at the factors of 24

i.e.  $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12$  and  $\pm 24$

By trial and error we substitute these values for  $x$  into  $f(x)$  until we find a value that gives a zero remainder, i.e.  $f(a) = 0$  and so  $(x - a)$  is a factor.

$$f(x) = x^3 + 5x^2 - 2x - 24$$

$$f(1) =$$

$$=$$

$\therefore (x - 1)$  IS/IS NOT a factor

$$f(-1) =$$

$$=$$

$\therefore (x + 1)$  IS/IS NOT a factor

$$f(2) =$$

$$=$$

$\therefore (x - 2)$  IS/IS NOT a factor

As before, we use long division to find the other factor(s).

**Exercise** Verify that

$$x^3 + 5x^2 - 2x - 24 = (x - 2)(x + 3)(x + 4)$$

## Example

Factorise  $f(x) = 2x^3 - 9x^2 + 7x + 6$ .

## Solution

1. Find the factors of 6
2. Find one factor by calculating  $f(a)$  where  $a$  is a factor of 6.
3. Find the remaining factor by long division

## Exercises on factoring polynomials

Exercise 1.12.4

page 39.