# Fundamentals Lecture Six

**Factorial $n$, Permutations. . .**

- Factorial $n$ $n!$
- Permutations ${}^4P_2$
- Combinations ${}^5C_3$
- The Binomial Theorem $(a+b)^n$
- Pascal's triangle

**Factors of Polynomials** Long division of polynomials.

# Expansions and Factors

$n!$ is the symbol for *factorial n*.

$$n! = n\,(n-1)\,(n-2)\,(n-3)\ldots 4.3.2.1$$

For example

$$4! = 4 \times 3 \times 2 \times 1 = 24,$$

$$6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$$

**Note**:

$$\text{(i)}\, n! = n\,(n-1)\,(n-2)\,(n-3)\ldots 4.3.2.1,$$

$$= n\,(n-1)!$$

$$\text{(ii)}\ n = \frac{n!}{(n-1)!}$$

$$\text{(iii)}\, 0! = 1 \text{(By definition)}.$$

# Examples

1. $5! = 5 \times (4 \times 3 \times 2 \times 1)$
   $= 5 \times 4!$
2. $\frac{6!}{5!} = \frac{6\times5!}{5!} = 6$

# Combinations and Permutations

$$^{n}C_r = \frac{n!}{r!\,(n-r)!}$$

$$^{n}P_r = \frac{n!}{(n-r)!}$$

# Examples

Evaluate the following expressions

1. $^{5}C_3$
2. $^{5}P_3$

(10)

(60)

# Pascal's Triangle

```
            1
          1   1
        1   2   1
      1   3   3   1
    1   4   6   4   1
  1   5  10  10   5   1
1   6  15  20  15   6   1
```

See p1-8 or p1-25 of MATH141 notes.

# Expansions

Standard binomial expansions of $(a+b)^n$ can be found using either Pascal's Triangle or the Binomial Theorem.

# Examples

$$(a+b)^0 = 1$$

$$(a+b)^1 = a+b$$

$$(a+b)^2 = a^2+2ab+b^2$$

$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$$

$$(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$$

**Question**: What is the relationship between the coefficients in these expansions and the numbers in Pascal's triangle?

**Note**: The '$a$' terms *decrease* in power from $n$ to 0 whilst the '$b$' terms *increase* in power from 0 to $n$.

# Exercise

1. Write down the expansion of $(a+b)^5$ using Pascal's triangle

2. Write down the expansion of $(1+x)^5$

$$a^5+5a^4b+10a^3b^2+10a^2b^3+5a^1b^4+b^5$$

$$1+5x+10x^2+10x^3+5x^4+x^5$$

# Example

Expand $(a-b)^4$.

Hint.

$$(a-b) = (a+(-b))\,.$$

$$(a-b)^4 =$$

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

$$a^4 + 4a^3\left(-b\right) + 6a^2\left(-b\right)^2 + 4a\left(-b\right)^3 + \left(-b\right)^4$$

$$a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$$

# Exercises on this stuff

Exercise 1.11.4.2
page 31.

For each of the questions do as many of the subquestions as you require in order to gain mastery of the basic technique.

# Long Division

Recall:

$1224 \div 8$ by ‘long division

$$8\,\big|\,\overline{1224}$$

$\therefore 1224 \div 8 = 153$ OR $8 \times 153 = 1224$

$$11\,\big|\,\overline{13268}$$

$\therefore 13268 \div 11 = 1206\frac{2}{11}$ OR
$11 \times 1206\frac{2}{11} = 13268$

# Examples

$$x - 4 \,\big|\, \overline{x^2 + 5x - 36}$$

$$\therefore x^2 + 5x - 36 =$$

Note that knowing one factor of the quadratic has allowed us to find the other using long division.

$$x + 1 \,\big|\, \overline{x^3 - 2x^2 - x + 2}$$

$$(x-4)(x+9)$$

$$\begin{aligned} x^3-2x^2-x+2 &= (x+1)\left(x^2-3x+2\right) \\ &= (x+1)(x-2)(x-1) \end{aligned}$$

In the following example

$$\left(4x^3 + 13x + 33\right) \div (2x + 3)$$

note that the cubic expression $4x^3 + 13x + 33$ has no $x^2$ term. We re-write

$$4x^3 + 13x + 33 = 4x^3 + \mathbf{0}x^2 + 13x + 33$$

$$2x + 3 \,\big|\, \overline{4x^3 + 0x^2 + 13x + 33}$$

$\therefore \left(4x^3 + 13x + 33\right) =$

$$(2x+3)\left(2x^2-3x+11\right)$$

$$x+1\,\overline{\big)\,x^3+\mathbf{0x^2}+\mathbf{0x}+1}$$

$$\therefore\; x^3+1=$$

$$x-3\,\overline{\big)\,3x^3-7x^2+2x+4}$$

$$\therefore\; \frac{3x^3-7x^2+2x+4}{x-3}=$$

OR $\quad 3x^3-7x^2+2x+4=$

$$\left(x + 1\right) \left(x^{2} - x + 1\right)$$

$$3x^2 + 2x + 8 + \frac{28}{x-3}$$

$$\left(3x^2 + 2x + 8\right)(x-3) + 28$$

# Exercise

Evaluate the following using long division.

1. $(a^3 + 6a^2 + 3a - 3) \div (a + 1)$
2. $(y^2 + 2y) \overline{\big) y^4 + 3y^3 + 3y^2 + 2y}$
3. $\frac{x^3+1}{x^3-1}$
4. $\frac{a^3+b^3}{a+b}$

# Solutions

1. $a^2 + 5a - 2$ remainder -1 OR
   $a^2 + 5a - 2 - \frac{1}{a+1}$

2. $y^2 + y + 1$

3. 1 remainder 2 or $1 + \frac{2}{x^3-1}$

4. $a^2 - ab + b^2$