

2.2 LIMITS

2.2.1 The basics

Definition: If $\lim_{x \rightarrow a} f(x) = L$, then as \underline{x} approaches \underline{a} , $\underline{f(x)}$ approaches \underline{L} , where \underline{L} is a finite number.

Example

Consider $f(x) = x + 2$. As x approaches 2, what happens to $f(x)$?

x	1.95	1.99	1.995	2	2.005	2.01	2.05
$f(x)$							

The table shows that, as x approaches 2 from either the left or right of 2, $f(x)$ approaches 4; i.e. we can make $f(x)$ as close as we like to 4 by making x sufficiently close to 2.

We write

$$\begin{aligned}\lim_{x \rightarrow 2} f(x) &= \lim_{x \rightarrow 2} (x + 2) \\ &= \underline{4}\end{aligned}$$

When looking at this limit we are not concerned what happens at $x = 2$ only what is happening to $f(x)$ as x approaches 2.

The limit of a function as x approaches a point does not depend on the value of the function at the point.

For example, consider the function

$$y = \begin{cases} x, & x \neq 2 \\ 4, & x = 2 \end{cases}$$

$$\lim_{x \rightarrow 2} f(x) = 2 \text{ but } f(2) = 4.$$

However, if $f(x)$ is a ‘nice’ function we can find $\lim_{x \rightarrow a} f(x)$ by substituting $x = a$ into $f(x)$.

Examples

1. $\lim_{x \rightarrow 2} (x + 2) = \underline{\quad}$

2. $\lim_{x \rightarrow -3} x^3 = \underline{\quad}$

4

-27

2.2.2 Theorems on Limits

If $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = M$, then

1. If $f(x) = k$, where k is a constant, then $\lim_{x \rightarrow a} f(x) = \underline{k}$.

For example, figure 4 shows the function $f(x) = 3$. Observe that regardless of the value of a , as

$x \rightarrow a$, $f(x) = 3$, therefore

$\lim_{x \rightarrow a} f(x) = 3$.

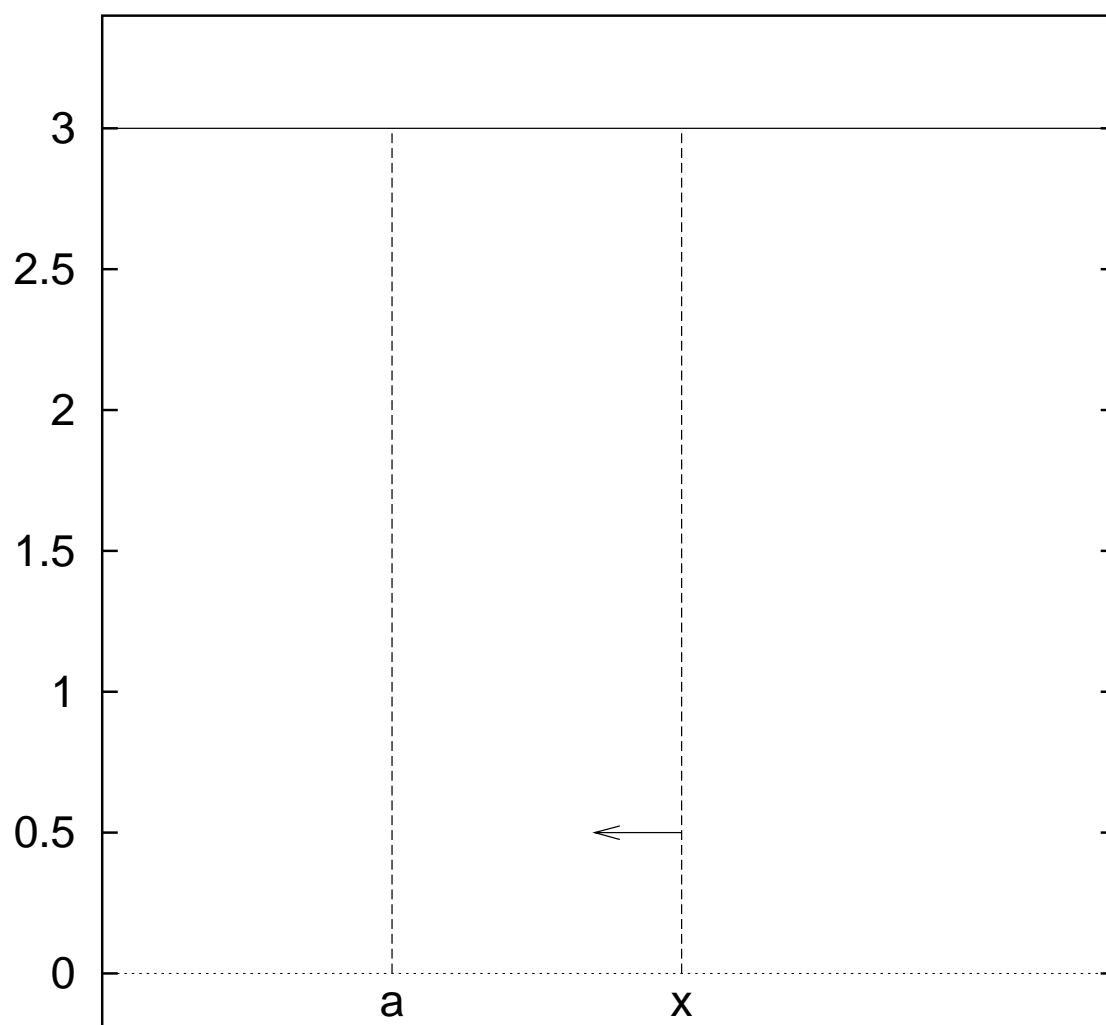


Figure 4: The function $f(x) = 3$.

2.

$$\begin{aligned}\lim_{x \rightarrow a} [f(x) \pm g(x)] &= \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x) \\ &= \underline{L \pm M}.\end{aligned}$$

For example

$$\begin{aligned}\lim_{x \rightarrow 2} (x^2 - 3x + 5) &= \lim_{x \rightarrow 2} x^2 - \lim_{x \rightarrow 2} 3x + \lim_{x \rightarrow 2} 5 \\ &= \\ &= \underline{3}\end{aligned}$$

$$(2)^2 - (3 \times 2) + 5$$

3.

$$\begin{aligned}\lim_{x \rightarrow a} [f(x) \cdot g(x)] &= \lim_{x \rightarrow a} f(x) \times \lim_{x \rightarrow a} g(x) \\ &= \underline{LM}.\end{aligned}$$

For example,

$$\begin{aligned}\lim_{x \rightarrow -1} 2x(x^2 - 4) &= \lim_{x \rightarrow -1} 2x \times \lim_{x \rightarrow -1} (x^2 - 4) \\ &= \\ &= \\ &= \underline{6}.\end{aligned}$$

$$2(-1) \times \left((-1)^2 - 4 \right)$$

$$-2 \times -3$$

4.

$$\begin{aligned} \lim_{x \rightarrow a} \left[\frac{f(x)}{g(x)} \right] &= \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} \\ &= \frac{L}{M} \quad \text{for } \underline{M \neq 0}. \end{aligned}$$

For example

$$\begin{aligned} \lim_{x \rightarrow 3} \left[\frac{x^2 + 2}{x + 1} \right] &= \frac{\lim_{x \rightarrow 3} (x^2 + 2)}{\lim_{x \rightarrow 3} (x + 1)} \\ &= \\ &= \underline{\quad} . \end{aligned}$$

$$\frac{3^2 + 2}{3 + 1} = \frac{11}{4}$$

2.2.3 Limits of Rational Functions

Definition A rational function is a function that can be expressed in the form $f(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials.

For example, $\frac{x^5 - 2x^2 + 1}{x^2 - 4}$, $\frac{x}{x + 1}$, $\frac{1}{x^5}$ etc.

Consider the following limit (using Theorem 4)

$$\begin{aligned} \lim_{x \rightarrow 2} \frac{5x^3 + 4}{x - 3} &= \frac{\lim_{x \rightarrow 2} 5x^3 + 4}{\lim_{x \rightarrow 2} x - 3} \\ &= \frac{\quad}{\quad} \\ &= \underline{\underline{-44}} \end{aligned}$$

Now consider $\lim_{x \rightarrow 1} \frac{1 - x^2}{1 - x}$. Theorem 4 is not applicable because $\lim_{x \rightarrow 1} (1 - x) = \underline{\underline{0}}$.

However, if both the numerator and denominator of a rational function approach zero as x approaches a , then the numerator and denominator will have a common factor $(x - a)$ and the limit can often be calculated by first cancelling the common factors.

For example,

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{1 - x^2}{1 - x} &= \frac{\lim_{x \rightarrow 1} (1 - x^2)}{\lim_{x \rightarrow 1} (1 - x)} \\ &= \\ &= \end{aligned}$$

Now

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{1 - x^2}{1 - x} &= \lim_{x \rightarrow 1} \frac{(1 - x)(1 + x)}{(1 - x)} \\ &= \lim_{x \rightarrow 1} (1 + x) \\ &= \lim_{x \rightarrow 1} (1 + x) \\ &= 2.\end{aligned}$$

Examples Find

1. $\lim_{x \rightarrow 3} \frac{x^2 - 6x + 9}{x - 3}$.

2. $\lim_{x \rightarrow -4} \frac{2x + 8}{x^2 + x - 12}$.

$$\begin{aligned}\lim_{x \rightarrow 3} \frac{x^2 - 6x + 9}{x - 3} &= \lim_{x \rightarrow 3} \frac{(x - 3)^2}{(x - 3)} \\ &= \lim_{x \rightarrow 3} (x - 3) \\ &= 0.\end{aligned}$$

$$\begin{aligned}\lim_{x \rightarrow -4} \frac{2x + 8}{x^2 + x - 12} &= \lim_{x \rightarrow -4} \frac{2(x + 4)}{(x + 4)(x - 3)} \\ &= \lim_{x \rightarrow -4} \frac{2}{x - 3} \\ &= \frac{-2}{7}\end{aligned}$$

If the denominator of a rational function has a limit of zero and the numerator has a limit that is not zero, then the limit of the rational function approaches infinity or we say that the limit does not exist.

For example $\lim_{x \rightarrow 4} \frac{1}{x - 4} = \frac{1}{0} \rightarrow \infty$ or

$\lim_{x \rightarrow 4} \frac{1}{x - 4}$ D.N.E..

2.2.4 Limit of Rational Functions as $x \rightarrow \pm\infty$

Consider the graphs of the polynomials x , x^2 , x^3 and x^4 . What happens when $x \rightarrow +\infty$ and when $x \rightarrow -\infty$?

We have

$$\lim_{x \rightarrow +\infty} x^n = +\infty \quad n = 1, 2, 3 \dots$$

$$\lim_{x \rightarrow -\infty} x^n = \begin{cases} +\infty, & n = 2, 4, 6 \dots \\ -\infty, & n = 1, 3, 5 \dots \end{cases}$$

Consider the limit $\lim_{x \rightarrow +\infty} \frac{3x + 5}{6x - 8}$.

Now, $\lim_{x \rightarrow +\infty} (3x + 5) = \underline{\underline{\infty}}$ and

$\lim_{x \rightarrow +\infty} (6x - 8) = \underline{\underline{\infty}}$ i.e. the numerator and denominator both approach infinity as x approaches infinity.

We can solve this type of limit by dividing each term in both the numerator and denominator by the highest power of x that occurs in the function.

In our example $\lim_{x \rightarrow +\infty} \frac{3x + 5}{6x - 8}$ the highest power of x that occurs is $x^1 = x$.

Therefore

$$\lim_{x \rightarrow +\infty} \frac{3x + 5}{6x - 8} =$$

=

=

Example

$$\lim_{x \rightarrow \infty} \frac{4x^2 - x}{2x^3 - 5} =$$

=

=

=

$$\frac{\frac{1}{x}}{\frac{1}{x}} \cdot \frac{3x + 5}{6x - 8}$$

$$\frac{\frac{3x}{x} + \frac{5}{x}}{\frac{6x}{x} - \frac{8}{x}}$$

$$\frac{3 + \frac{5}{x}}{6 - \frac{8}{x}} = \frac{1}{2}$$

$$\frac{\frac{1}{x^3}}{\frac{1}{x^3}} \cdot \frac{4x^2 - x}{2x^3 - 5}$$

$$\frac{\frac{4x^2}{x^3} - \frac{x}{x^3}}{\frac{2x^3}{x^3} - \frac{5}{x^3}}$$

$$\frac{\frac{4}{x} - \frac{1}{x^2}}{2 - \frac{5}{x^3}}$$

$$\frac{0 - 0}{2 - 0} = 0.$$

2.2.5 Exercises

pages 11 & 12.

2.2.6 Revision of key ideas

The following questions are about the key ideas in this section.

1. What do we mean by the expression

$$\lim_{x \rightarrow a} f(x) = L?$$

2. What are the following limits?

(a) $\lim_{x \rightarrow 0} x^n$ ($n > 0$)

(b) $\lim_{x \rightarrow +\infty} x^n$ ($n > 0$)

(c) $\lim_{x \rightarrow -\infty} x^n$ ($n > 0$)

(d) $\lim_{x \rightarrow 0} \frac{1}{x}$