2.10 IMPLICIT DIFFERENTIATION

2.10.1 The Chain Rule Revisited

Recall definition.
If g is differentiable at x and f is differentiable at $g(x)$, then
$[f(g(x)]' = \underline{f'(g(x))} \cdot g'(x)$.

If we introduce another variable, say u, we can write the chain rule in another form.

Given $y = f(g(x)$, let $u = \underline{g(x)}$ and so $y = \underline{f(u)}$.

Now, if $y = f(u)$, then $\dfrac{dy}{du} = \underline{f'(u)}$ and if $u = g(x)$, then $\dfrac{du}{dx} = \underline{g'(x)}$.

If we use these substitutions then the derivative of $y = f(g(x)$, where $u = g(x)$, is

$$\frac{dy}{dx} = f'(g(x).\, g'(x)$$

$$= \underline{f'(u) \cdot \frac{du}{dx}} \quad \text{let } u = g(x)$$

$$= \underline{\frac{dy}{du} \cdot \frac{du}{dx}}$$

2.10.2 Alternative Definition of the Chain Rule.

If g is differentiable at x and f is differentiable at $g(x)$ and also if $y = \underline{f\left(g\left(x\right)\right)}$ and $u = \underline{g\left(x\right)}$ then $y = \underline{f\left(u\right)}$ and

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \underline{\frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}}.$$

This formula is easy to remember if we note that the LHS is exactly what we get if we 'cancel' the du's on the right.

2.10.2.1 Exercises on the chain rule

1. Differentiate:

   a. $y = (3x+5)^4$ b. $w = \sqrt{4+3\sqrt{t}}$

   c. $y = \cos(\cos\theta)$ d. $y = \sqrt{x}\tan^3(\sqrt{x})$

2. In each of the following find $\frac{\mathrm{d}y}{\mathrm{d}x}$ by first making an appropriate choice for u.

   a. $y = \left(x^3+2x\right)^{37}$ b. $y = \left(x^3 - \frac{7}{x}\right)^{-2}$

   c. $y = \sin\left(\frac{1}{x^2}\right)$ d. $y = \cos^3(\sin 2x)$

1a. Differentiate $y = (3x+5)^4$.

$$\text{Let,} \quad u = 3x + 5. \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 3$$

$$\begin{aligned}
\text{Then,} \quad y &= u^4, \\
\frac{\mathrm{d}y}{\mathrm{d}u} &= 4u^3 \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x} \\
&= 4u^3 \cdot 3 \\
&= 12\,(3x+5)^3
\end{aligned}$$

1b. Differentiate $w = \sqrt{4 + 3\sqrt{t}}$.

$$
\begin{aligned}
\text{Let,} \quad u &= 4 + 3\sqrt{t}. \\
\frac{\mathrm{d}u}{\mathrm{d}t} &= \frac{3}{2}t^{-1/2} \\
\text{Then,} \quad w &= u^{1/2}, \\
\frac{\mathrm{d}w}{\mathrm{d}u} &= \frac{1}{2}u^{-1/2} \\
\frac{\mathrm{d}w}{\mathrm{d}t} &= \frac{\mathrm{d}w}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}t} \\
&= \frac{1}{2}u^{-1/2} \cdot \frac{3}{2}t^{-1/2} \\
&= \frac{3}{4} \cdot t^{-1/2} \cdot \left(4 + 3\sqrt{t}\right)^{-1/2} \\
&= \frac{3}{4\sqrt{t} \cdot \sqrt{4 + 3\sqrt{t}}}
\end{aligned}
$$

1c. Differentiate $w = \cos(\cos\theta)$

$$\begin{aligned}
\text{Let,} \quad u &= \cos\theta. \\
\frac{\mathrm{d}u}{\mathrm{d}\theta} &= -\sin\theta \\
\text{Then,} \quad w &= \cos u, \\
\frac{\mathrm{d}w}{\mathrm{d}u} &= -\sin u \\
\frac{\mathrm{d}w}{\mathrm{d}\theta} &= \frac{\mathrm{d}w}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}\theta} \\
&= -(\sin u) \cdot (-\sin\theta) \\
&= \sin(\cos\theta) \cdot \sin\theta
\end{aligned}$$

1d. Differentiate $y = \sqrt{x}\,\tan^3(\sqrt{x})$.

Let, $u = (x)^{1/2}$.

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{2}(x)^{-1/2}$$

$$y = u\tan^3 u,$$

$$\frac{\mathrm{d}y}{\mathrm{d}u} = \tan^3 u + u\frac{\mathrm{d}}{\mathrm{d}u}\tan^3 u$$

$$= \tan^3 u + 3u\tan^2 u\sec^2 u(*)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\cdot\frac{\mathrm{d}u}{\mathrm{d}x}$$

$$= \left(\tan^3 u + 3u\tan^2 u\sec^2 u\right)\cdot\frac{1}{2}(x)^{-1/}$$

(*) I missed out some calculations here.
You should be able to do this!

$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\tan^3(\sqrt{x}) + 3\sqrt{x}\tan^2(\sqrt{x})\cdot\sec^2(\sqrt{x})}{2\sqrt{x}}$$

2a. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ by first making an appropriate choice for u.

$y = \left(x^3 + 2x\right)^{37}$

$$\begin{aligned}
\text{Let,} \quad u &= x^3 + 2x \\
\frac{\mathrm{d}u}{\mathrm{d}x} &= 3x^2 + 2 \\
\text{Then,} \quad y &= u^{37} \\
\frac{\mathrm{d}y}{\mathrm{d}u} &= 37u^{36} \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x} \\
&= 37u^{36} \cdot \left(3x^2 + 2\right) \\
&= 37\left(3x^2 + 2\right)\left(x^3 + 2x\right)^{36}
\end{aligned}$$

2b. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ by first making an appropriate choice for u.

$$y = \left(x^3 - \frac{7}{x}\right)^{-2}$$

$$\begin{aligned}
\text{Let,} \quad u &= x^3 - \frac{7}{x} \\
\frac{\mathrm{d}u}{\mathrm{d}x} &= 3x^2 + 7x^{-2} \\
\text{Then,} \quad y &= (u)^{-2} \\
\frac{\mathrm{d}y}{\mathrm{d}u} &= -2\,(u)^{-3} \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x} \\
&= -2\,(u)^{-3} \cdot \left(3x^2 + 7x^{-2}\right) \\
&= -2\left(3x^2 + 7x^{-2}\right)\left(3x^3 - 7x^{-1}\right)^{-3}
\end{aligned}$$

2c. find $\frac{\mathrm{d}y}{\mathrm{d}x}$ by first making an appropriate choice for u.

$$y = \sin\left(\frac{1}{x^2}\right)$$

$$\begin{aligned}
\text{Let,} \quad u &= x^{-2} \\
\frac{\mathrm{d}u}{\mathrm{d}x} &= -2x^{-3} \\
\text{Then,} \quad y &= \sin u \\
\frac{\mathrm{d}y}{\mathrm{d}u} &= \cos u \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x} \\
&= (\cos u) \cdot -2x^{-3} \\
&= -2x^{-3} \cos\left(x^{-2}\right)
\end{aligned}$$

2d. Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ by first making an appropriate choice for u.

$y = \cos^3(\sin 2x)$.

$$\text{Let,} \quad u = \sin 2x \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 2\cos 2x$$

$$\text{Then,} \quad y = \cos^3 u$$

We need to find $\dfrac{\mathrm{d}y}{\mathrm{d}u}$. We can't do this directly so we make a second substitution

$$\text{Let,} \quad w = \cos u \qquad \frac{\mathrm{d}w}{\mathrm{d}u} = -\sin u$$

$$y = w^3 \qquad \frac{\mathrm{d}y}{\mathrm{d}w} = 3w^2$$

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}u} &= \frac{\mathrm{d}y}{\mathrm{d}w} \cdot \frac{\mathrm{d}w}{\mathrm{d}u} \\ &= 3w^2 \cdot -\sin u \\ &= -3\left(\sin u\right)\left(\cos^2 u\right) \end{aligned}$$

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x} \\
&= -3\left(\sin u\right)\left(\cos^2 u\right) \cdot 2\cos 2x \\
&= -6\left(\sin\left[\sin 2x\right]\right)\left(\cos^2\left[\sin 2x\right]\right) \cdot \cos 2x
\end{aligned}
$$

You are expected to work through section 2.11.2 before the lecture.

2.10.3 Implicit Differentiation

So far our equations have generally been expressed in explicit form. For example,

$y = 3x - 5$,

$s = -16t^2 + 20t$ and $u = 2w - w^2$ where

y is explicitly in terms of $\underline{x}$,

s is explicitly in terms of $\underline{t}$ and

u is explicitly in terms of $\underline{w}$.

What happens if we are unable to write y in terms of x, for example

$x^2 - 2y^3 + 4y = 2$

How do we find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in this example?

To find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ we use implicit differentiation, in which we assume that y is a differentiable function of x.

To find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ we differentiate each term with respect to x. When we differentiate a term involving x alone, we can differentiate as usual. But when we differentiate a term involving y we must apply the Chain Rule.

In general,

$$\frac{\mathrm{d}}{\mathrm{d}x}[g(y)] = \underline{\frac{\mathrm{d}}{\mathrm{d}y}[g(y)] \cdot \frac{\mathrm{d}y}{\mathrm{d}x}}$$
$$= \underline{g'(y)\frac{\mathrm{d}y}{\mathrm{d}x}}$$

Example Differentiate the following with respect to x:

1. $3x^2$
2. $2y^3$
3. $x + 3y$
4. xy^2

1. Differentiate with respect to x. $3x^2$

2. Differentiate with respect to x. $2y^3$

$$\frac{\mathrm{d}}{\mathrm{d}x} = 6x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(2y^3\right) = \frac{\mathrm{d}}{\mathrm{d}y}\left(2y^3\right)\frac{\mathrm{d}y}{\mathrm{d}x}$$
$$= 6y^2 \cdot \frac{\mathrm{d}y}{\mathrm{d}x}$$

3. Differentiate with respect to x.

$x + 3y$

4. Differentiate with respect to x. xy^2

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(x+3y) &= \frac{\mathrm{d}}{\mathrm{d}x}x + \frac{\mathrm{d}}{\mathrm{d}x}3y \\
&= 1 + \frac{\mathrm{d}}{\mathrm{d}y}(3y) \cdot \frac{\mathrm{d}y}{\mathrm{d}x} \\
&= 1 + 3\frac{\mathrm{d}y}{\mathrm{d}x}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(xy^2\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(y^2\right) \cdot x + \frac{\mathrm{d}}{\mathrm{d}x}(x) \cdot y^2 \\
&= \frac{\mathrm{d}}{\mathrm{d}y}\left(y^2\right) \cdot \frac{\mathrm{d}y}{\mathrm{d}x} \cdot x + y^2 \\
&= 2yx \cdot \frac{\mathrm{d}y}{\mathrm{d}x} + y^2
\end{aligned}
$$

Given an **equation** involving x and y we can find $\frac{\mathrm{d}y}{\mathrm{d}x}$ as follows.

1. Differentiate all terms of the equations with respect to x.
2. Collect all terms involving $\frac{\mathrm{d}y}{\mathrm{d}x}$ on the left hand side of the equation and move all other terms to the right side.
3. Take $\frac{\mathrm{d}y}{\mathrm{d}x}$ out as a factor on the left hand side of the equation.
4. Solve for $\frac{\mathrm{d}y}{\mathrm{d}x}$ by dividing both sides of the equation by the left hand factor that multiplies $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Examples

1. find $\frac{\mathrm{d}y}{\mathrm{d}x}$ given $x^2 + y^2 - 3x - 6y = -5$. $\left(\frac{3-2x}{2y-6}\right)$
2. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ given $x^2 + xy + 3y^2 = 4$. $\left(\frac{-2x-y}{x+6y}\right)$
3. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ given $5y^2 + \sin y^2 = x^2$. $\left(\frac{x}{y[5+2\cos y^2]}\right)$
4. Use implicit differentiation to find $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ if $4x^2 - 2y^2 = 9$. $\left(\frac{-9}{y^3}\right)$
5. Show, using implicit differentiation, that $x(x^2 + 3y^2) = c$ is the solution to the differential equation $x^2 + y^2 + 2xy\frac{\mathrm{d}y}{\mathrm{d}x} = 0$.

1. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ given
$x^2 + y^2 - 3x - 6y = -5.$

$$\frac{\mathrm{d}}{\mathrm{d}x}x^2 + \frac{\mathrm{d}}{\mathrm{d}x}y^2 - \frac{\mathrm{d}}{\mathrm{d}x}3x - \frac{\mathrm{d}}{\mathrm{d}x}6y = \frac{\mathrm{d}}{\mathrm{d}x}(-5)$$

$$2x + \frac{\mathrm{d}}{\mathrm{d}y}\left(y^2\right) \cdot \frac{\mathrm{d}y}{\mathrm{d}x} - 3 - 6\frac{\mathrm{d}}{\mathrm{d}y}(y) \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$\begin{aligned} 2x - 3 + 2y \cdot \frac{\mathrm{d}y}{\mathrm{d}x} - 6\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ (2y-6)\frac{\mathrm{d}y}{\mathrm{d}x} &= 3 - 2x \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{3-2x}{2y-6} \end{aligned}$$

2. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ given $x^2 + xy + 3y^2 = 4$.

$$\frac{\mathrm{d}}{\mathrm{d}x}x^2 + \frac{\mathrm{d}}{\mathrm{d}x}(xy) + \frac{\mathrm{d}}{\mathrm{d}x}\left(3y^2\right) = \frac{\mathrm{d}}{\mathrm{d}x}4$$

$$2x + x\frac{\mathrm{d}}{\mathrm{d}x}(y) + y\frac{\mathrm{d}}{\mathrm{d}x}(x) + \frac{\mathrm{d}}{\mathrm{d}y}\left(3y^2\right) \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$\begin{aligned} 2x + x\frac{\mathrm{d}y}{\mathrm{d}x} + y + 6y\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ (x + 6y)\frac{\mathrm{d}y}{\mathrm{d}x} &= -(y + 2x) \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{-(y + 2x)}{x + 6y} \end{aligned}$$

3. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ given $5y^2 + \sin y^2 = x^2$.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(5y^2\right) + \frac{\mathrm{d}}{\mathrm{d}x}\left(\sin y^2\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}y}\left(5y^2\right) \cdot \frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\mathrm{d}}{\mathrm{d}y}\left(\sin y^2\right) \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = 2x.$$

We need to calculate $\frac{\mathrm{d}}{\mathrm{d}y}\left(\sin y^2\right)$. Consider

$$z = \sin y^2$$

$$\text{Let,} \quad w = y^2 \qquad \frac{\mathrm{d}w}{\mathrm{d}y} = 2y$$

$$z = \sin w \qquad \frac{\mathrm{d}z}{\mathrm{d}w} = \cos w$$

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}y} &= \frac{\mathrm{d}z}{\mathrm{d}w} \cdot \frac{\mathrm{d}w}{\mathrm{d}y} \\ &= (\cos w) \cdot (2y) \\ &= 2y \cos\left(y^2\right)\end{aligned}$$

$$10y\frac{\mathrm{d}y}{\mathrm{d}x} + 2y\cos\left(y^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$$
$$\left[10y + 2y\cos\left(y^2\right)\right]\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x}{10y + 2y\cos(y^2)}$$
$$= \frac{x}{y\left[5 + \cos(y^2)\right]}$$

4. Use implicit differentiation to find $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ if $4x^2 - 2y^2 = 9$.

First of all we must find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left(4x^2\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left(2y^2\right) &= \frac{\mathrm{d}}{\mathrm{d}x}(9) \\ 8x - \frac{\mathrm{d}}{\mathrm{d}y}\left(2y^2\right) \cdot \frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ 8x - 4y \cdot \frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{8x}{4y} \\ &= \frac{2x}{y} \end{aligned}$$

Now we calculate $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ using the *quotient rule*

$$
\begin{aligned}
\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{y\frac{\mathrm{d}}{\mathrm{d}x}(2x) - 2x\frac{\mathrm{d}}{\mathrm{d}x}(y)}{y^2} \\
&= \frac{2y - 2x\frac{\mathrm{d}}{\mathrm{d}y}(y)\cdot\frac{\mathrm{d}y}{\mathrm{d}x}}{y^2} \\
&= \frac{2y - 2x\frac{\mathrm{d}y}{\mathrm{d}x}}{y^2} \\
&= \frac{2y - 2x\frac{2x}{y}}{y^2} \\
&= \frac{2y\frac{y}{y} - 2x\frac{2x}{y}}{y^2} \\
&= \frac{2y^2 - 4x^2}{y^3} \\
&= \frac{-\left(4x^2 - 2y^2\right)}{y^3} \\
&= \frac{-9}{y^3}
\end{aligned}
$$

Show, using implicit differentiation, that $x(x^2+3y^2)=c$ is the solution to the differential equation
$x^2+y^2+2xy\dfrac{\mathrm{d}y}{\mathrm{d}x}=0.$

We need to differentiate the equation $x(x^2+3y^2)=c$

$$\begin{aligned}
x^3+3xy^2 &= c\\
\frac{\mathrm{d}}{\mathrm{d}x}\left(x^3\right)+3\frac{\mathrm{d}}{\mathrm{d}x}\left(xy^2\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(c\right)\\
3x^2+3y^2\frac{\mathrm{d}}{\mathrm{d}x}\left(x\right)+3x\frac{\mathrm{d}}{\mathrm{d}x}\left(y^2\right) &= 0\\
3x^2+3y^2+3x\frac{\mathrm{d}}{\mathrm{d}y}\left(y^2\right)\cdot\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\
3x^2+3y^2+3x\cdot(2y)\cdot\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\
\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{\left(x^2+y^2\right)}{2xy}
\end{aligned}$$

We now substitute our expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ into the differential equation

$$\begin{aligned} x^2 + y^2 + 2xy\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ x^2 + y^2 + 2xy - \frac{(x^2+y^2)}{2xy} &= 0 \\ x^2 + y^2 - x^2 - y^2 &= 0. \end{aligned}$$

2.10.4 Revision Questions

The folowing questions are about the key ideas in this section.

1. Suppose that $y = f(u)$ where $u = g(x)$. What is $\dfrac{\mathrm{d}y}{\mathrm{d}x}$?
2. Given the *implicit* equation $f(y, x) = 0$ write down the procedure to find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

2.10.5 Exercises

Hint. There's always an exam question on implicit differentiation.