DIFFERENTIATION

2.1 FUNCTIONS

2.1.1 The Basics

Definition: A function f is a rule which, for each number x in some subset $A \in \mathbb{R}$, assigns a (unique) number $y = f(x) \in \mathbb{R}$.

Definition: The set A is the set of all possible x-values of the function f and is called the domain of f. The domain of f is denoted by $\operatorname{Dom} f$.

Definition: The set of all possible y-values of the function f is called the range of f. The range of f is denoted by $\operatorname{Range} f$.

Examples

1. Given $f(x) = x^2$.
   Dom $f = \mathbb{R}$ or $\underline{(-\infty, \infty)}$ or $\underline{\{x : x \in \mathbb{R}\}}$
   Range $f = \mathbb{R}^+$ or $\underline{[0, \infty)}$ or $\underline{\{y : y \geq 0\}}$
2. Given $g(x) = x^2$. $x \geq 0$
   Dom $g = \mathbb{R}^+$
   Range $g = \mathbb{R}^+$
3. Given $h(x) = \frac{1}{x-3}$.
   Dom $h = \{x : x \neq 3\}$ or $(-\infty, 3) \cup (3, \infty)$ or $\mathbb{R} - \{3\}$.
   Range $h = \{y : y \neq 0\}$ or $(-\infty, 0) \cup (0, \infty)$ or $\mathbb{R} - \{0\}$.

Note that for the functions f and h, we choose the domain to be the largest set of x for which the function 'makes sense'. This is the natural domain of f and h.

Definition: Two functions f and g are equal if and only if $f(x) = g(x)$ and $\mathrm{Dom}\, f = \mathrm{Dom}\, g$.

For example, consider the two functions $f(x) = \sqrt{x^2 - 1}$ and $g(x) = \sqrt{x^2 - 1}$ $x \geq 1$.

The (natural) domain of f is $x^2 - 1 \geq 0$ i.e. $x^2 \geq 1 \Rightarrow |x| \geq 1$. i.e.
$\mathrm{Dom}\, f = \{x : x \leq -1 \text{ or } x \geq 1\}$.

The domain of g is given as
$\mathrm{Dom}\, g = x \geq 1$.

Therefore $\mathrm{Dom}\, f \neq \mathrm{Dom}\, g$ and so the two functions f and g are not equal.

Definition: Given a function $y = f(x)$, $\underline{x}$ is called the independent variable since it may be chosen freely from Dom f and $\underline{y}$ is called the dependent variable since its value depends on the value chosen for x.

2.1.2 Functions defined Piecewise

Sometimes a function may be defined by formulas that have been 'pierced together'.

The absolute value function, $f(x) = |x|$ can be written in an equivalent piecewise form as

$$f(x) = \begin{cases} x, & \underline{x \geq 0} \\ -x, & \underline{x < 0} \end{cases}$$

2.1.3 Graph of a Function

Definition: The graph in the xy-plane ($\mathbb{R}^2$) of a function f is defined to be the equation of the graph $y = f(x)$.

Examples Sketch the graphs of the following functions.

1. $f(x) = x + 2$.
2. $g(x) = \begin{cases} x+2, & x \neq 2 \\ 6, & x = 2. \end{cases}$
3. $h(x) = \dfrac{x^2 - 4}{x - 2}$.

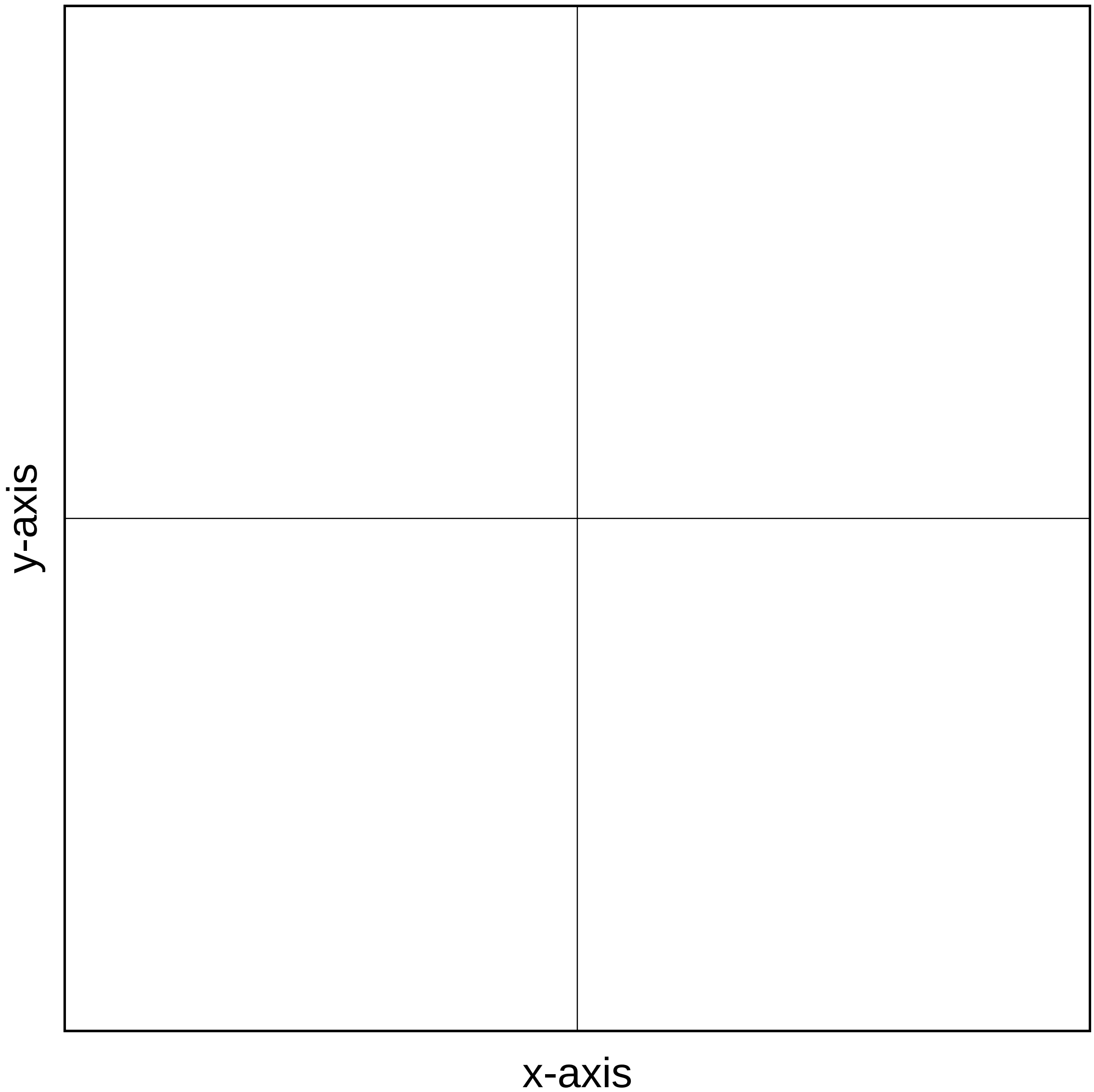

Figure 1: (a)$f(x) = x + 2$

$$g(x) = \begin{cases} x+2, & x \neq 2 \\ 6, & x = 2. \end{cases}$$

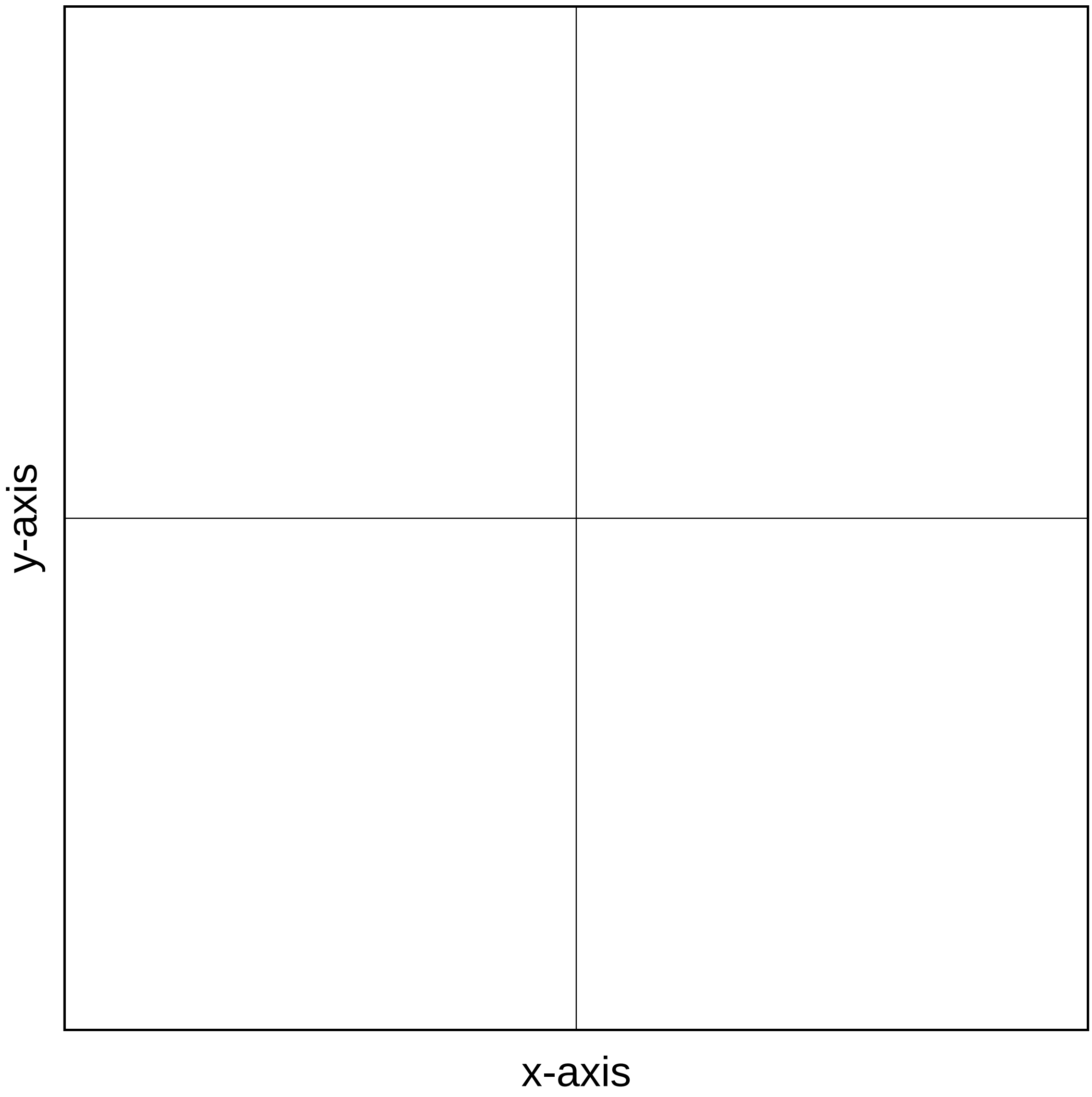

Figure 1: (b) $g(x)$

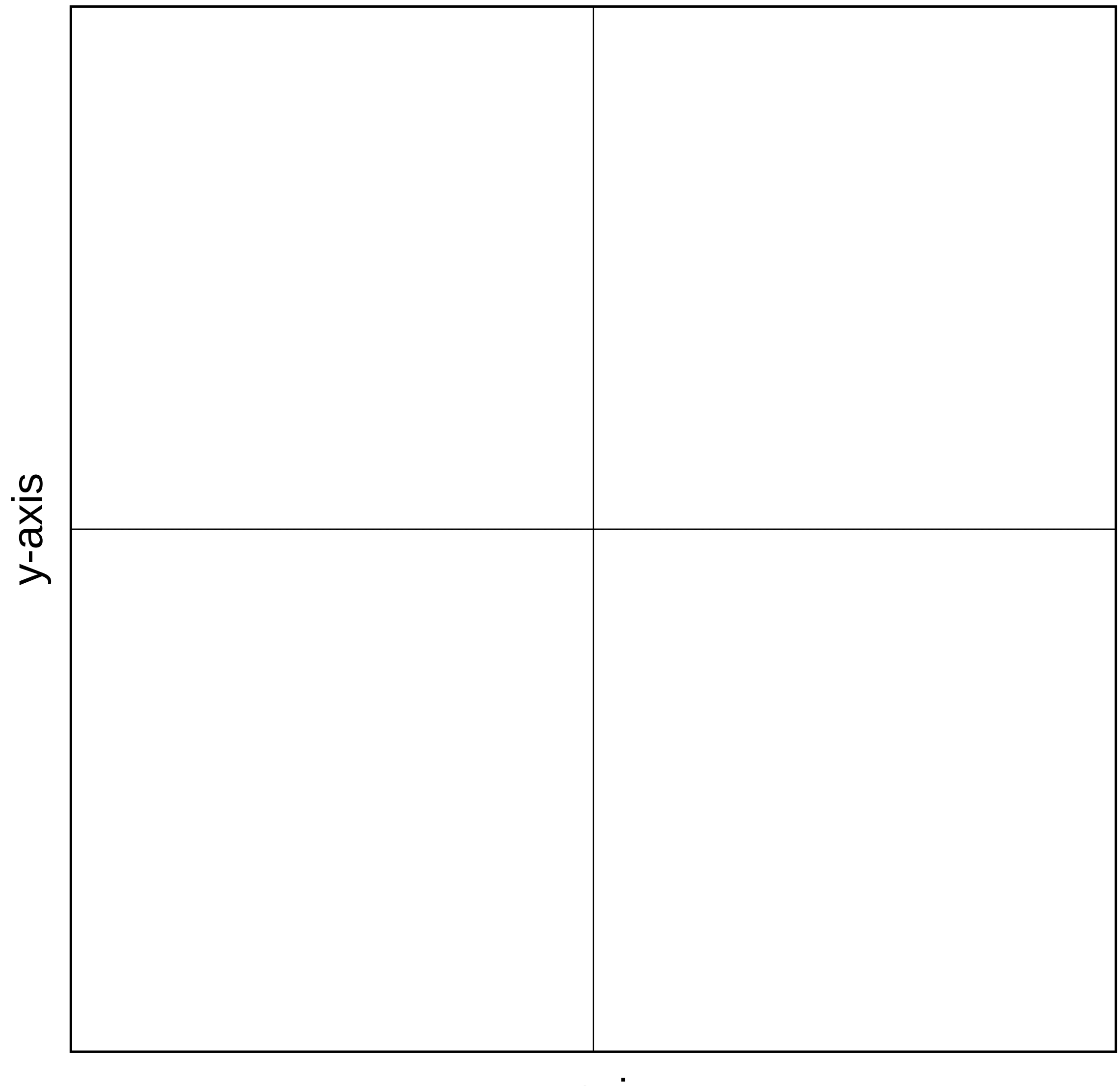

Figure 1: (c)$h(x) = \dfrac{x^2 - 4}{x - 2}$

Important points to consider when sketching the graph of $y = f(x)$ are:

1. Find Dom f (in particular, note where there are "gaps" in Dom f, and observe how $y = f(x)$ behaves near the endpoints of the gaps),
2. Find the obvious zeroes (if any) of the function, and
3. Determine how f behaves as x becomes large and positive, and as x becomes large in absolute value, but negative.

In addition, if the graph of $y = f(x)$ is known, then the graphs of each of the functions $y = f(x - a)$, $y = f(x) + a$ and $y = \frac{1}{f(x)}$ are easy to determine:

1. The graph of $y = f(x - a)$ has the same shape as $y = f(x)$, but is translated $|a|$ units to the right (for $a > 0$), or to the left (for $a < 0$).

2. The graph of $y = f(x) + a$ has the same shape as $y = f(x)$, but is translated $|a|$ units up (for $a > 0$), or down (for $a < 0$).

3. The graph of $y = \frac{1}{f(x)}$ has the same algebraic sign as $f(x)$, but is undefined when $f(x) = 0$, and approaches 0 whenever $|f(x)|$ becomes large.

Examples

1. Sketch $y = |x - 3| + 2$.

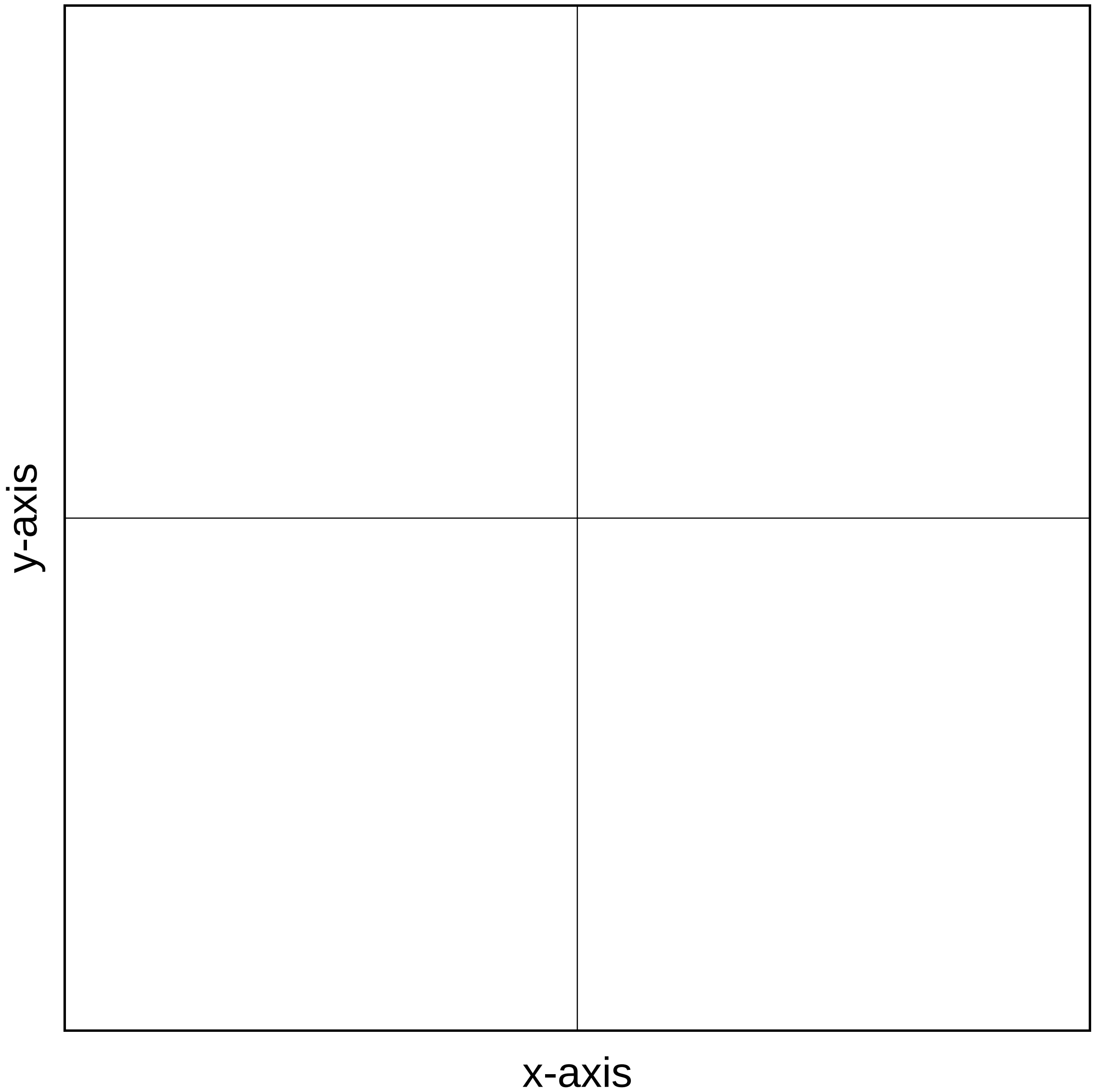

Figure 2: (a)

2. To find the domain and range of a function, it can sometimes be easier to sketch the function and then read off the domain and range.

Sketch the graph of $f(x) = \sqrt{x-2}$ and hence determine $\operatorname{Dom} f$ and $\operatorname{Range} f$.

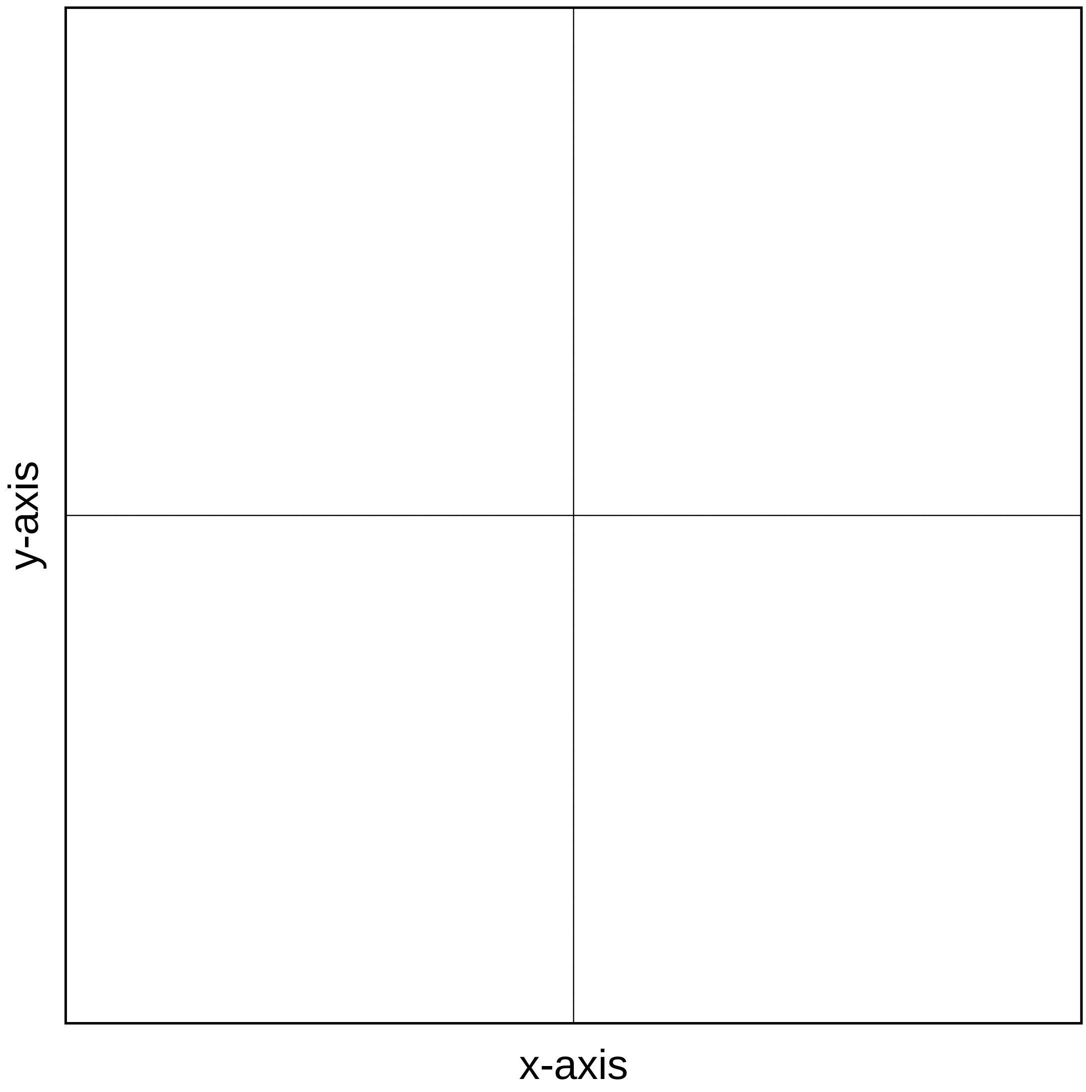

Figure 3: (b) $f(x) = \sqrt{x-2}$

2.1.4 Exercises

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2.1.5 Operations with Functions

If f and g are functions, and $a \in \mathbb{R}$, then

1. $(f \pm g)(x) = \underline{f(x) \pm g(x)}$, for x in both $\operatorname{Dom} g$ and $\operatorname{Dom} f$.
2. $(fg)(x) = \underline{f(x)g(x)}$, for x in both $\operatorname{Dom} g$ and $\operatorname{Dom} f$.
3. $\dfrac{f}{g}(x) = \underline{\dfrac{f(x)}{g(x)}}$ for x in both $\operatorname{Dom} f$ and $\operatorname{Dom} g$, and $g(x) \neq 0$.
4. $(af)(x) = \underline{af(x)}$, for $x \in \operatorname{Dom} f$.

Examples

1. Let $f(x) = 1 + \sqrt{x-2}$ and $g(x) = x - 1$. Find $(f+g)(x)$ and its domain.
2. Let $f(x) = \sqrt{2-x}$ and $g(x) = \sqrt{1+x}$. Find $(f-g)(x)$ and its domain.
3. Let $f(x) = \sqrt{x}$ and $g(x) = 3\sqrt{x}$. Find $(fg)(x)$ and its domain.
4. Let $f(x) = \sqrt{x-5}$ and $g(x) = x + 3$. Find (f/g) and its domain.

(1) Let $f(x) = 1 + \sqrt{x-2}$ and $g(x) = x - 1$.
Find $(f+g)(x)$ and its domain.

$(f \pm g)(x) = f(x) \pm g(x)$, for x in both Dom g and Dom f.

$$\text{Dom}\, f = \{x : x \geq 2\}$$

$$\text{Dom}\, g = \{x : x \in \Re\}$$

$$\begin{aligned} \text{Dom}\, f + g &= \{x : x \geq 2\} \cap \{x : x \in \Re\} \\ &= \{x : x \geq 2\} \end{aligned}$$

$(f + g)(x) = \sqrt{x - 2} + x$ with
$\text{Dom}(f + g)(x) = \{x : x \geq 2\}$

(2) Let $f(x) = \sqrt{2-x}$ and $g(x) = \sqrt{1+x}$.

Find $(f-g)(x)$ and its domain.

$(f \pm g)(x) = f(x) \pm g(x)$, for x in both $\operatorname{Dom} g$ and $\operatorname{Dom} f$.

$$\text{Dom}\, f = \{x : x \leq 2\}$$

$$\text{Dom}\, g = \{x : x \geq -1\}$$

$$\begin{aligned}\text{Dom}\, f - g &= \{x : x \leq 2\} \cap \{x : x \geq -1\} \\ &= \{x : -1 \leq x \leq 2\}\end{aligned}$$

$(f - g)(x) = \sqrt{x - 2} - \sqrt{1 + x}$ with
$\text{Dom}(f - g)(x) = \{x : -1 \leq x \leq 2\}$

(3) Let $f(x) = \sqrt{x}$ and $g(x) = 3\sqrt{x}$.
Find $(fg)(x)$ and its domain.

$(fg)(x) = f(x)g(x)$, for x in both $\operatorname{Dom} g$ and $\operatorname{Dom} f$.

Note: $\operatorname{Dom}(fg) \neq \operatorname{Dom}(3x) = \mathbb{R}$

$$\operatorname{Dom} f = \{x : x \geq 0\}$$

$$\operatorname{Dom} g = \{x : x \geq 0\}$$

$$\begin{aligned} \operatorname{Dom}(fg)(x) &= \{x : x \geq 0\} \cap \{x : x \geq 0\} \\ &= \{x : x \geq 0\} \end{aligned}$$

$(fg)(x) = 3x$ with

$\operatorname{Dom}(fg)(x) = \{x : x \geq 0\}$

(4) Let $f(x) = \sqrt{x-5}$ and $g(x) = x - 7$. Find $f/g(x)$ and its domain.

$\dfrac{f}{g}(x) = \dfrac{f(x)}{g(x)}$ for x in both $\operatorname{Dom} f$ and $\operatorname{Dom} g$, and $g(x) \neq 0$.

$$\text{Dom}\, f = \{x : x \geq 5\}$$

$$\text{Dom}\, g = \{x : x \in \Re\}$$

$$\{x : g\,(x) = 0\} = \{x = 7\}$$

$$\begin{aligned} \text{Dom}\, \frac{f}{g} &= \{x : x \geq 5\} \cap \{x : x \in \Re\} \\ &\quad - \{x = 7\} \\ &= \{x : x \in [5, 7) \cup (7, \infty)\} \end{aligned}$$

$$(f/g)(x) = \frac{\sqrt{x-5}}{x-7} \text{ with}$$

$$\text{Dom}(f/g)(x) = \{x : x \in [5, 7) \cup (7, \infty)\}$$

Composition of Functions.

Definition: Given functions f and g, the composition of f with g, denoted by $f \circ g$, is the function defined by $(f \circ g)(x) = f(g(x))$.

The domain of the composition of f with g is given by

$$\text{Dom}(f \circ g) = \{x \in \text{Dom}\, g \quad \text{and} \quad g(x) \in \text{Dom}\, f\}.$$

1. Note: $f \circ g$ is sometimes described as 'a function of a function'.

The domain of the composition of f with g is given by

$$\underline{\text{Dom}\,(f \circ g) = \{x \in \text{Dom}\, g \quad \text{and}}$$
$$\underline{g\,(x) \in \text{Dom}\, f\}}.$$

2. Note: The domain may seem complicated but it is only sensible. To evaluate $f(g(x))$, <u>x must be in the domain of g</u> to evaluate to evaluate $g(x)$, and also <u>$g\,(x)$ must be in the domain of f</u> to evaluate $f(g(x))$.

Examples

1. Find (i) $f \circ g$ and (ii) $g \circ f$ for each of the following.
   (a) $f(x) = 2x, g(x) = x^2 + 1$.
   (b) $f(x) = 3x - 2, g(x) = |x|$.
   (c) $f(x) = \sqrt{x+1}, g(x) = x - 2$.
2. Find $(f \circ g)(x)$ if $f(x) = x^2 + 3$ and $g(x) = x - 1$. State the domain of $f \circ g$.
3. Find $(f \circ g)(x)$ and its domain if $f(x) = \sqrt{x}$ and $g(x) = x - 1$.
4. Find $(f \circ g)(x)$ if $f(x) = x^2 + 3$ and $g(x) = \sqrt{x}$. State the domain of $f \circ g$.
5. Express $h(x) = \sqrt{4 - 3x}$ as a composition of two functions $f(x)$ and $g(x)$ such that $h(x) = f(g(x))$.

2.1.6 Exercises

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2.1.7 Revision of key ideas

The following questions are about the key ideas in this section.

1. Suppose that $y = f(x)$. What it is meant by *the domain of the function* f and by *the range of the function* f?
2. Suppose that $a = f(b)$. Which is the *independent variable* and which is the *dependent variable*?

3. Given the graph $y = f(x)$ explain how it is related to the graphs: (i) $y = f(x - 1)$, (ii) $y = f(x) + a$ and (iii) $y = 1/f(x)$.

4. Given two functions f and g what do we mean by the *composition of f with g*? What is the definition of Range $f \circ g$ and Dom $f \circ g$?