

# MATH111 – Spring 2007

## Tutorial Sheet – Week 11

This tutorial sheet covers sections 13.1–13.4 of the notes (constant yield harvesting).

### Chapter 13. Harvesting

#### Revision of Key Ideas

The following questions are about the key ideas in this chapter that relate to constant yield harvesting.

1. The logistic differential equation is

$$x' = rx \left(1 - \frac{x}{K}\right).$$

Write down the logistic differential equation with constant yield harvesting.

2. Under what conditions does ‘constant yield harvesting’ commonly occur?  
 3. Determine the maximum sustainable rate of harvesting in the logistic differential equation with constant yield harvesting,

$$x' = rx \left(1 - \frac{x}{K}\right) - H.$$

4. In this question we consider the logistic differential equation with constant yield harvesting

$$x' = rx \left(1 - \frac{x}{K}\right) - H. \tag{1}$$

- (a) Determine the steady-state solutions of equation (1).  
 (b) Determine the stability of the steady-state solutions of equation (1).  
 Hint. Use a graphical method.  
 (c) Sketch the steady-state diagram for equation (1) showing stable solutions and unstable solutions with solid and dashed lines respectively.  
 (d) Discuss the implications of your steady-state diagram.
5. The population density of a certain kind of fish is modelled by the differential equation

$$\dot{x} = f(x),$$

where the function  $f(x)$  is a continuous function satisfying the following properties:

- $f(0) = 0$  and  $f(K) = 0$ .
- If  $0 < x < K$  then  $f(x) > 0$ .
- If  $x > K$  then  $f(x) < 0$ .

- (a) Sketch the function  $f(x)$ .  
 (b) Identify the steady-state solutions on your sketch and determine their stability.  
 (c) The fish is subjected to constant yield harvesting so that the model equation becomes

$$\dot{x} = f(x) - H.$$

On your sketch identify the critical value of  $H$ ,  $H_{\text{cr}}$ , above which harvesting is unsustainable.

## Exercises

Questions 1 & 3 are from the 2004 week 12 assignment. Question 2 is from the 2006 exam paper.

1. A population of sandhill cranes (*Grus canadensis*) has been modelled by a logistic equation with carrying capacity of 194,600 members and intrinsic growth rate  $0.0987\text{year}^{-1}$ . Find the critical harvest rate for which constant yield harvesting will drive the population to extinction, and find the equilibrium population size under constant yield harvesting of 3000 birds per year. You may quote appropriate results from your lecture notes.
2. (a) The growth of a tumour inside the human body can be represented by the equation

$$\frac{dT}{dt} = \beta T \left(1 - \frac{T}{K}\right), \quad T(0) = T_0.$$

where  $T$  is the size of the tumour,  $\beta$  denotes the growth rate of the tumour and  $K$  is the maximum tumour size.

- (i) Sketch the rate of change of tumour growth  $\dot{T}$  as a function of  $T$ .
  - (ii) Using your sketch describe how the long-term evolution of the differential equation depends upon the choice of the initial condition  $T_0$ .
  - (iii) Suggest biomedical interpretations for the steady-state solutions  $T = 0$  and  $T = K$ .
- (b) The growth of a tumour inside the human body when radiation therapy is used can be represented by the equation

$$\frac{dT}{dt} = \beta T \left(1 - \frac{T}{K}\right) - \mathcal{I}, \quad T(0) = T_0,$$

where the parameter  $\mathcal{I}$  is proportional to the intensity of the radiation.

- (i) Find the steady-state solutions of this model and sketch how they vary as a function of the intensity. (Do not calculate stability).
  - (ii) Hence, or otherwise, determine a condition for the tumour to be destroyed.
  - (iii) Suppose that for a particular patient  $K = 1000$  and  $\beta = 2$  (in appropriate units). Suppose that the value of the irradiance  $\mathcal{I}$  can be controlled with an error tolerance of  $\pm 1\%$ . Suggest a value for  $\mathcal{I}$  to destroy the tumour, justifying your answer.
3. (Tricky) This question is *much* harder than anything you will be given in an exam. Only attempt it if you think you are a top student.

Consider the model (Smith, 1963)

$$x' = \frac{rx(K-x)}{K+ax}$$

subjected to constant yield harvesting

$$x' = \frac{rx(K-x)}{K+ax} - h. \quad (2)$$

- (a) Consider the equation

$$dx^2 + ex + f = 0, \quad d > 0 \quad f > 0.$$

Explain why the roots of this equation, should they exist, are positive if and only if  $e < 0$ .

- (b) Show that the steady-states of equation (2) are given by

$$rx^{*2} + (ah - rK)x^* + hK = 0$$

Explain why harvesting is only sustainable if

- $ah - rK < 0$ .
  - $a^2h^2 - 2r(a+2)Kh + r^2K^2 \geq 0$ .
- (c) Suppose that  $r = K = 1$  and  $a = 2$ . Find the maximum sustainable value of the harvesting parameter  $h$ .