MATH111 – Spring 2007
Tutorial Sheet – Week 8

This tutorial sheet covers chapters 10 of the notes.

Chapter 10. Applications of First-Order Differential Equations

Exercises

Pollution modelling in the river Thames

This exercise is based upon a workshop run by Professor G.C. Wake[1] on "Mathematics in the Environment: Modelling Pollution" at the International Conference on Mathematical Modelling and Computation (5-8th June2006, University Brunei Darussalam). The workshop was based upon a series of mathematical models developed to analyse pollution in the River Thames (London, England) in the 1950s and 1960s. This is the simplest such model.

1. **First 'Pond' Model**. We consider a one-off dump of pollutant (P) into a 'pond', or sewage plant. A side effect of dumping pollutant is that it removes oxygen from the pond. This scenario is represented by the model

$$\frac{\mathrm{d}P}{\mathrm{d}t} = -k_p P, \quad P(t=0) = P_0, \tag{1}$$

$$\frac{\mathrm{d}O_2}{\mathrm{d}t} = -k_o P, \quad O_2(t=0) = 100\%. \tag{2}$$

   In the absence of pollutant the oxygen in the pond is in equilibrium with atmospheric oxygen. The initial of 100% represents the equilibrium value, in which the oxygen has been scaled so that it has no units.

   (a) What does it mean in terms of the pond if there is a value of t such that $O_2(t) = 0$?

   (b) Solve the model. **Hint**. First solve equation (1) to find $P(t)$. Then substituted this expression into equation (2).

   (c) Sketch a graph showing how the concentration of oxygen in the pond varies as a function of time.

   (d) It is known that there is a threshold oxygen concentration in the pond, such that if the level of oxygen in the pond decreases below 30% that the fish in the pond will die. We must develop a decision-support procedure for the model.

      (i) Show that there is a critical value of P_0, P_{cr} such that if $P_0 \leq P_{\mathrm{cr}}$ then $\lim_{t\to\infty} O_2(t) \geq 30\%$.

      (ii) When $P_0 > P_{\mathrm{cr}}$ find the time, t_{cr}, it takes for the concentration of oxygen in the pond to reach 30%.

   (e) What are suitable units for the following quantities

      (i) pollutant concentration (P)?

      (ii) time (t)?

      (iii) the level of oxygen in the pond?

      (iv) the rate constant k_p?

      (v) the rate constant k_o?

In producing the final model other factors incorporated into the model included:

- The position of the sewage plant along the River Thames.

[1]Professor G.C Wake, Centre for Mathematics in Industry, Massey University, Auckland, New Zealand

- The effect of tidal flow.
- Different dumping strategies.

Drug absorption through a skin patch

The model discussed in this section is analysed in the paper.

Kubota, K., and Maibach, H.I. (1992). A Compartment Model for Percutaneous Absorption: Compatibility of Lag Time and Steady-State Flux with Diffusion Model. *Journal of Pharmaceutical Sciences* **81**(9), 863–865.

A drug delivery vehicle, such as a skin patch, is attached to a patient. The amount of drug per unit area (X) in the skin is described by the following differential equation

$$\frac{\mathrm{d}X}{\mathrm{d}t} = -\left(k_{12} + k_{1B}\right) X + k_{21} A_0, \quad X0 = 0. \tag{3}$$

In this equation A_0 is the concentration of drug per unit area in the delivery device, which is maintained at a constant level. The parameters k_{12} and k_{21} are the rate at which drug enters the vehicle from the skin and the skin from the vehicle respectively. The parameter k_{1B} is the rate at which drug leaves the skin and enters the blood stream.

1. Solve equation (3) to find the amount of drug per unit area in the skin as a function of time.
2. The cumulative amount of drug per unit area excreted from the skin (Ae_s) is expressed by

$$Ae_s = \int k_{1B} X \mathrm{d}t, \quad Ae_s\left(0\right) = 0.$$

   Determine Ae_s.

A population model

The Gompertz equation is derived from the differential equation pair

$$\begin{aligned} \frac{\mathrm{d}N}{\mathrm{d}t} &= \gamma N, \quad N\left(0\right) = N_0 \\ \frac{\mathrm{d}\gamma}{\mathrm{d}t} &= -\alpha\gamma, \quad \gamma\left(0\right) = \gamma_0. \end{aligned} \tag{4}$$

This system describes a population growing an an initial exponential rate of γ_0. The growth rate itself is time dependent and decays exponentially with rate α. As the derivative approaches zero in the first equation, the theoretical population converges to a limiting value. The theoretical limit is termed the carrying capacity.

(a) Solve system (4) to find $N\left(t\right)$.

Hint. First solve the second equation to find $\gamma\left(t\right)$. Then substitute this into the first equation. Solve the resulting differential equation to find $N\left(t\right)$.

(b) Find the limiting population size: $\lim_{t\to\infty} N\left(t\right)$.

This question is inspired by the article.

S. Michelson, A.S. Glicksman, and J.T. Leith. Growth in solid heterogeneous human colon adenocarcinomas: comparison of simple logistical models. *Cell and Tissue Kinetics*, **20**:343–355, 1987.