School of Mathematics & Applied Statistics

MATH111: Applied Mathematical Modelling

Assignment Week 9 Spring 2007

Student Name: ____ *Student Number:* ________________

FULL WORKING is to be shown for all solutions.
Untidy or badly set out work will not be marked.
This assignment is to be handed in at the end of your during the Wednesday lecture of week 10.

Assignment Guidelines

You are expected to structure your assignments as if writing a report for presentation to people unfamiliar with the work covered by the assignment.

Your report should be structured as follows:

1. introductory remarks
   - state purpose of assignment; and
   - state proposed tasks.
2. discussion of theory (if appropriate)
3. discussion of results
   - present outputs of each task;
   - analyse outputs of each task, including
     - comment on what your mathematical results mean in terms of the underlying physical problem.
     - comment on unexpected results;
4. concluding remarks: state whether purpose of assignment was achieved. You should answer the following questions regarding this assignment.
   - What was the most important thing that you learnt? Why was it 'important'?
   - What was the most puzzling thing that you did? (If nothing was puzzling, say so!)
5. Bibliography (if required)
6. Appendices
   - MAPLE program(s), containing comment lines explaining the purpose of your code.
   - MAPLE outputs where you think they are required to further amplify comments you have made in your report. Do *not* include every output you generated.
7. If you are not certain what is required in your report you should speak to the lecturer before you hand it in. If you don't ask, don't whinge if you lose marks because you didn't do what you should have done.

Continued on next page

School of Mathematics & Applied Statistics **MATH111**: *Applied Mathematical Modelling*
Assignment Week 9 Spring 2007 Submission Receipt

Student Name: ____ *Student Number*: ________________

Tutorial Class: ___________ *Date Submitted:* ________________ *Tutor Initials:* ________________

Graphs and tables should be included at appropriate locations in the body of the report or as appendices at the end of the report. Please ensure all handwritten work is tidy and legible and that every page is present and in the intended order.

The grade a student receives will be the lab demonstrator's subjective assessment of how much effort that student seems to have put into creating their report. Please note that missing or incomplete outputs, inadequate discussions, and/or poor presentation will result in a low grade even if you have successfully completed all the assigned tasks. Note that marks to questions/tasks (if provided) is only indicative.

Here are some good ways to *lose* marks (5% for each one):

- No title ('MATH111 Assignment Week x' is not a satisfactory title).
- No introduction.
- No theory.
- No sections/section headings.
- Not including the model equations.
- Not discussing the model equations.
- No conclusions or summary.
- No figures.
- Inadequate referencing of sources.
- No appendices (if required).
- No page numbers.
- Repeating the questions in your report and answering them. You're supposed to write a report!
- Using the question/task numbers in your report. These don't make sense to a reader who hasn't read the assignment sheet.
- Including every graph you generated during your investigation. Summarise your findings where appropriate!
- Poor graphs: no title, no labels, too small, too large, not numbering figures etc.
- Stating that your graph uses colour, such as 'blue' and 'black' lines, but only providing a black and white graphic.
- Poor quality output: difficult to read; pages out of order.
- Not showing signs of having carried out further reading when you have been asked to read specific article(s). (**-10%**)

This list is *not* exhaustive.

Instructions

You should work you way through this assignment, answering questions and making notes where appropriate. Where appropriate you should adapt Maple programs that you have used in previous lab sessions.

You will find it very *useful* to save any programs that you write onto a disk which you bring to subsequent labs.

1. Use a text editor such as NotePad (Programs/Accessories/NotePad) to write your program.
2. Save your program onto a disk (or alternatively onto the C drive) as a *text* file.
3. To load your program into Maple enter `read "A:/file";` where `file` is the name of your program.
4. If your program generates an error message:
   (a) Enter the command `restart;` into Maple.
   (b) Look at your code for syntax errors. Correct the code and reload it.
   (c) If you can't find your error, ask for assistance.

Solving first-order autonomous differential equations in Maple

1 Background

In this assignment you will learn how to numerically solve a first-order autonomous differential equation using Maple. Your new skill will be used to investigate the properties of the spruce budworm model. The spruce budworm model was introduced in Ludwig *et al* (1978). You will need to refer to this article to complete this assignment.

D. Ludwig, D.D. Jones & C.S. Holling. 1978. Qualitative analysis of insect outbreak systems: the spruce budworm and forest. *Journal of Animal Ecology*, **47**, 315-332.

This article is available as an e-reading through the library catalogue. You can also access the article by finding the journal in library.

1.1 Insect outbreak model: Spruce budworm

The spruce budworm is a major problem in Canada where it can, with ferocious efficiency, defoliate the balsam fir. Ludwig *et al* (1978) modelled the population dynamics of the budworm population by the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = rx\left(1 - \frac{x}{q}\right) - \frac{x^2}{1+x^2}. \tag{1}$$

In equation 1 the parameter r is proportional to the birth rate of the budworm and the q is proportional to the density of foliage available on the trees. The term $-\frac{x^2}{1+x^2}$ models predation by birds. The quantitative form of this function is biologically important. When x is small ($0 < x \ll 1$) predation is small, because the birds tend to seek food elsewhere. When x is large ($x \gg 1$) predation reaches a constant value ($\frac{x^2}{1+x^2} \approx 1$) because, although the birds spend all their time eating the budworm, there are only a finite number of birds.

Question 1 *Is equation* (1) *linear or non-linear? Justify your answer.*

Question 2 (Read 'Level 1' of the article)

1. *What are the main features that limit the budworm population?*
2. *Explain in more detail why predation is small when x is small.*
3. *The predation function used in the paper (which is equivalent to the function used in the assignment) is given by*

$$g(B) = \beta\frac{B^2}{\alpha^2 + B^2}.$$

This functional response has a special name. What is it?

2 Theory

2.1 Steady-state solutions

They first step to understanding the long-term dynamics of a first-order nonlinear *differential* equation is to find the *steady-state solutions* of the equation. For the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x)$$

the steady-state solutions are the values of x^* such that

$$f(x^*) = 0.$$

(The second step is to determine their *stability*).

Question 3 *What are the first two steps to understand the long-term dynamics of a first-order nonlinear difference equation?*

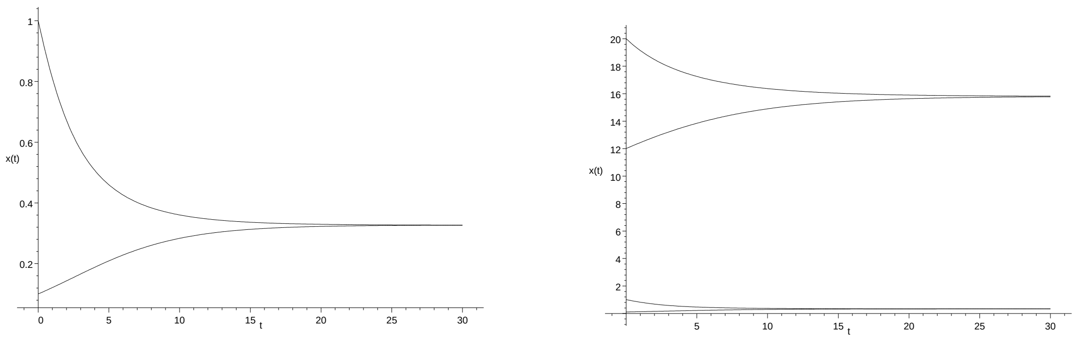
2.2 Graphical techniques

The concept of steady-state solutions for differential equations has not yet been introduced in lectures. However, in writing your report you are encourage to think about what your results might mean in terms of steady-state solutions. You should also think about what 'stability' might mean for steady-state solutions.

For a first-order *differential* equation it is often very useful to plot the function $y = f(x)$. From such a plot you can readily identify the values of x for which: $\frac{\mathrm{d}x}{\mathrm{d}t} > 0$, $\frac{\mathrm{d}x}{\mathrm{d}t} < 0$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$.

3 Assignment

Your first task is to download the code given in appendix A. When you run the code you should produce three figures, two of which are shown in figure 1. It should be obvious to you how to generalise the code to have more than two initial conditions.



(a) Initial conditions: $x(0) = 0.1$ and $x(0) = 1.0$

(b) Initial conditions: $x(0) = 0.1$, $x(0) = 1.0$, $x(0) = 12.0$, $x(0) = 20.0$

Figure 1: The numerical solution $x(t)$ to the budworm differential equation $x' = 0.3x(1 - x/20.0) - x^2/(1 + x^2)$.

3.1 Coding issues

In the code the parameter `step` controls the accuracy of the numerical solution. If this number is 'too large' then your numerical solution will be inaccurate. If it is too small then it will take too long for your code to run. You should note that the code provided in the appendix is not very sophisticated. A different Maple code would be used to investigate more realistic problems.

As usual we are mainly interested in the long-term behaviour of the population. The parameter `tend` controls how far into the 'future' the code looks.

Three *important* points to remember:

1. You will have to vary the value for `tend`. The 'right' value will depend upon the parameter values in the model.
2. If your graphs are not smooth, containing noticeably straight lines, then the value of `step` is too large. Divide it by two and try again.
3. You should check that any results you provide in your report are not too sensitive to the value of `step` by repeating your calculations halving the value for `step`.

3.2 Tasks

1. In the budworm model the long-term behaviour of the population depends upon the initial condition. This is *not* the case for the budworm model. Figure 1 shows two distinct behaviours: the long-term behaviour tends to either a 'small' value or a 'large value'.
   (a) By changing the initial conditions find a number a (to one decimal place) such that if $0 < x(0) < a$ the long term behaviour is towards the smaller value whereas if $a < x(0)$ the long term behaviour is towards the larger value
   (b) Which value for the long-term value of $x(t)$ do you think is preferred by a forest managers?
   (c) Does your answer to the previous question suggest a pest-control strategy?
2. Change the value for r in your code to $r = 0.1$. Try a variety of initial conditions. Report your findings and suggest a biological interpretation. (You may need to change the value of `tend`).
3. Change the value for r in your code to $r = 0.6$. Try a variety of initial conditions. Report your findings and suggest a biological interpretation. (You may need to change the value of `tend`).
4. Consider the output from the following Maple command
   `plot(0.3*x*(1-x/20.0)-x**2/(1+x**2),x=0..20);`
   (a) Compare the output from this command to your answer to question 1 and comment.
   (b) Change the value of r in the plot command to $r = 0.1$. Compare the new output to your answer to question 2 and comment. (You may find it useful to change the x range and possible to instruct Maple to use a y range).
   (c) Change the value of r in the plot command to $r = 0.6$. Compare the new output to your answer to question 3 and comment.
5. Show that the steady-state solutions of the budworm differential equation
   $$\frac{dx}{dt} = rx\left(1 - \frac{x}{q}\right) - \frac{x^2}{1+x^2}$$
   are $x = 0$ and the solutions, if any, of the equation
   $$r\left(1+x^2\right)\left(1 - \frac{x}{q}\right) - x = 0. \tag{2}$$
   Consider the output from the following Maple command
   `solve(0.3*(1+x**2)*(1-x/20)-x=0,x);`

(a) What is the maximum number of steady-state solutions that equation (2) can have?

(b) Compare the output from this command to your answers to questions 1 & 4a and comment.

(c) Change the value of r in the solve command to $r = 0.1$. Compare the new output to your answers to questions 2 & 4b and comment.

(d) Change the value of r in the solve command to $r = 0.6$. Compare the new output to your answers to questions 3 & 4c and comment.

(e) There is an important difference between the solutions when $r = 0.3$ to those when either $r = 0.1$ or $r = 0.6$. What is it? By decreasing the value of r from 0.3 find a critical value of r, $r_{\mathrm{cr},1}$ (to two decimal places). By increasing the value of r from 0.3 find a critical value of r, $r_{\mathrm{cr},2}$ (to one decimal place).

The 'nature' of the solutions when either $0 < r < r_{\mathrm{cr},1}$ should be the same as when $r > r_{\mathrm{cr},2}$. This 'nature' should be different to that when $r_{\mathrm{cr},2} < r < r_{\mathrm{cr},2}$. You should explain what is meant by 'nature' in mathematical terms.

6. A steady-state diagram shows how the steady-state solutions to a problem change as a parameter in the model is varied. A steady-state diagram for the equation

$$r\left(1+x^2\right)\left(1-\frac{x}{q}\right)-x=0,$$

with $q = 20$ can be generated by the following Maple commands.

```
r := 'r';
with(plots);
eqn := r*(1+x**2)*(1-x/20.0)-x;
implicitplot(eqn,r=0..1,x=0..20,labels=["birth rate r","steady-state solution x"]);
```

You might also like to replace the last command with

```
implicitplot(eqn,r=0..1,x=0..20,grid=[40,40]);
```

Comment on the biological implications of you plot. In particular, you should be able to explain almost all of your answers to the preceding questions using this figure.

7. Generate a steady-state diagram with $q = 5$. How does your steady-state diagram differ from the one with $q = 20$? By generating steady-state diagrams for different values of q you should be able to find a critical value of q, q_{cr}, (to one decimal place) at which the features of the steady-state diagram change significantly.

Question 4 *The spruce budworm model can be improved by introducing two additional differential equations. One for S and one for E. What do the variables S and E represents?*

4 Further reading

The conclusion that you write for this assignment should be informed by the content of the paper by Ludwig *et al* (1978). You should make it clear what you have learnt from reading this article.

5 Marking

Every student starts with a mark of 100. The questions and tasks for this assignment are worth a certain number of marks. Every time your answer to a question or task is incomplete or wrong you lose marks. In addition to losing marks in this way can also lose marks for a badly written report. There is no upper bound on the number of marks you can lose. If you make 17 bad mistakes (see the list on page two) then you will lose 85 marks. However, your mark will not be reduced below zero.

A Maple code

You do not *need* to retype this code. You can download it from

`http://www.uow.edu.au/~mnelson/teaching.dir/math111.dir/code.dir/budworm.html`

```
# budworm.maple  Maple program to solve a first-order
# 17.09.03       ordinary differential equation.
#
# NOTE. This is NOT the maple code that one would use to
# investigate a research problem, but it's good enough for the
# present purpose.
with(DEtools):

step := 0.1:  # this number controls how accurate the numerical
              # solution is.
tstart := 0:  # the initial value of time.
tend   := 30:  # the final value of time.

ic1 := [0,0.1];       # one initial condition in the form (t0, x(t0));
# two initial conditions both in the form (t0, x(t0));
ic2 := [0,0.1],[0,1.0];
# four initial conditions
ic3 := [0,0.1],[0,1.0],[0,12.0],[0,20.0];

r   := 0.3;   # budworm `birth-rate'.
q   := 20.0;   # `foilage density'.

# define the differential equation.  Note that we have to TELL maple
# that x is a function of time by writing x(t)
de1 := diff(x(t),t) = r*x(t)*(1-x(t)/q) -x(t)**2/(1+x(t)**2);

# calculate a solution trajectory from an initial condition.
DEplot(de1,x(t),t=tstart..tend,[ic1],stepsize=step,arrows=NONE, \
       linecolor=BLACK);

# compare solution trajectories from TWO initial conditions.
DEplot(de1,x(t),t=tstart..tend,[ic2],stepsize=step,arrows=NONE, \
       linecolor=BLACK);

# compare solution trajectories from FOUR initial conditions.
DEplot(de1,x(t),t=tstart..tend,[ic3],stepsize=step,arrows=NONE, \
       linecolor=BLACK);
```