

MID-YEAR EXAMINATION 2002

Course Number: AMAT.2110

Course Name: Discrete Dynamics

Question 5. (20 marks)

(a) The logistic difference equation is

$$x_{n+1} = rx_n(1 - x_n).$$

(i) Show that the fixed points of the logistic difference equation are $x^* = 0$ and $x^* = (r - 1)/r$.

[2 marks]

(ii) Determine the stability of the fixed points as a function of r and explain the biological implications of your answer.

[6 marks]

(iii) How does the solution to the logistic equation change as the parameter r is increased from 2.9 to 3.1?

[1 mark]

(b) Consider the difference equation

$$x_{n+1} = 0.1 + x_n^2.$$

(i) Find the fixed points of this map and determine their stability.

[3 marks]

(ii) If $x_0 = 0.8$ explain what happens as $n \rightarrow \infty$.

[2 mark]

(iii) If $x_0 = 0.9$ explain what happens as $n \rightarrow \infty$.

[2 mark]

(c) Consider the difference equation

$$x_{n+1} = f(x_n),$$

(i) What do we mean when we say that the pair x_0^* and x_1^* is a period-2 orbit of the function f ?

[1 mark]

(ii) Write down the conditions for the pair x_0^* and x_1^* to be stable and unstable. (*Remember to carefully define all the terms that appear in your solution*).

[3 marks]

Question 5. (13 marks)

Consider the logistic equation with fixed harvesting

$$x_{n+1} = rx_n(1 - x_n) - h,$$

where the positive quantities r and h are the static birth rate and the number of animals that are harvested respectively.

- (a) Show that the fixed points of the above difference equation are given by

$$x^* = \frac{(r - 1) \pm \sqrt{(1 - r)^2 - 4rh}}{2r}.$$

[2 marks]

- (b) Hence show that there is a critical value of the harvesting parameter, $h = h_{cr}$, such that if $h > h_{cr}$ harvesting is not sustainable. Identify this value h_{cr} .

[3 marks]

- (c) Suppose that $r = 2$ and $h = 0.1$.

- (i) Sketch the graph $y = rx(1 - x) - h$.

[1 mark]

- (ii) Using your graph explain why the population will become extinct if the initial value (x_0) is either 'too small' or 'too large'.

[2 marks]

- (iii) Locate the values 'too small' and 'too large' on your graph.

[1 mark]

- (d) Suppose that x represents the population of scallops in a particular location and that $r = 2$. A company decides to harvest the scallops using fixed harvesting. The profit \mathcal{P} that the company makes is given by the formulae

$$\mathcal{P} = h - 0.06.$$

The company asks you to recommend a value for its annual harvest. What would you suggest to the company? (Give reasons for your suggestions.)

[4 marks]