

## 6 Harvesting

### 6.1 Introduction

Consider an animal species that is harvested or hunted, e.g. deer, fish or rabbits. In this chapter, we study two harvesting, or hunting, strategies for this population.

Although the basic ideas are very simple, discrete models have been used in fishery management for some considerable time.

They have proven to be very useful in evaluating harvesting strategies with a view to both optimising the economic yield and to maintaining it.

During this chapter you will:

1. Analyse simple models for the management of harvested species.
  - Learn what is meant by a *fixed* harvesting strategy.
  - Learn what is meant by a *proportional* harvesting strategy.
2. Relate *mathematical* results to the *physical problem*.
  - Understand how a fixed harvesting strategy can drive a population to extinction.
  - Understand how a proportional harvesting strategy can drive a population to extinction.

2. Relate *mathematical* results to the *physical problem*.

- Learn how to optimise the yield from both the fixed and proportional harvesting strategies.
- Understand why proportional harvesting is a better strategy to manage a sustainable resource than fixed harvesting.
- To appreciate why in practice fixed harvesting might be preferred to proportional harvesting.

In this chapter we will assume that the population model in the absence of harvesting is the logistic equation.

Recall that the unscaled logistic equation is

$$N_{n+1} = N_n \left( r - \frac{N_n}{K} \right).$$

The scaled logistic equation is given by

$$x_{n+1} = rx_n(1 - x_n), \quad (6.1)$$

where  $x_n = \frac{N_n}{rK}$ . In studying the management of a renewable resource (the animal species) we assume that in equation (6.1)  $1 < r \leq 4$ .

**Question 6.1** *Why do we assume that  $1 < r \leq 4$  in equation (6.1)?*

We will typically consider the case  $r = 1.8$ . Before investigating how harvesting effects the dynamics of our model we should first determine what happens when there is no harvesting ( $h = 0$ ).

### Question 6.2

1. Show that when  $r = 1.8$  the fixed points of equation (6.1) are  $x^* = 0$  and  $x^* = \frac{0.8}{1.8}$ .
2. Show that the eigenvalues are  $\lambda = 1.8$  when  $x^* = 0$  and  $\lambda = 0.2$  when  $x^* = \frac{0.8}{1.8}$ .
3. Using figure 6.1 determine what happens to the population size  $x_n$  as  $n \rightarrow \infty$ .

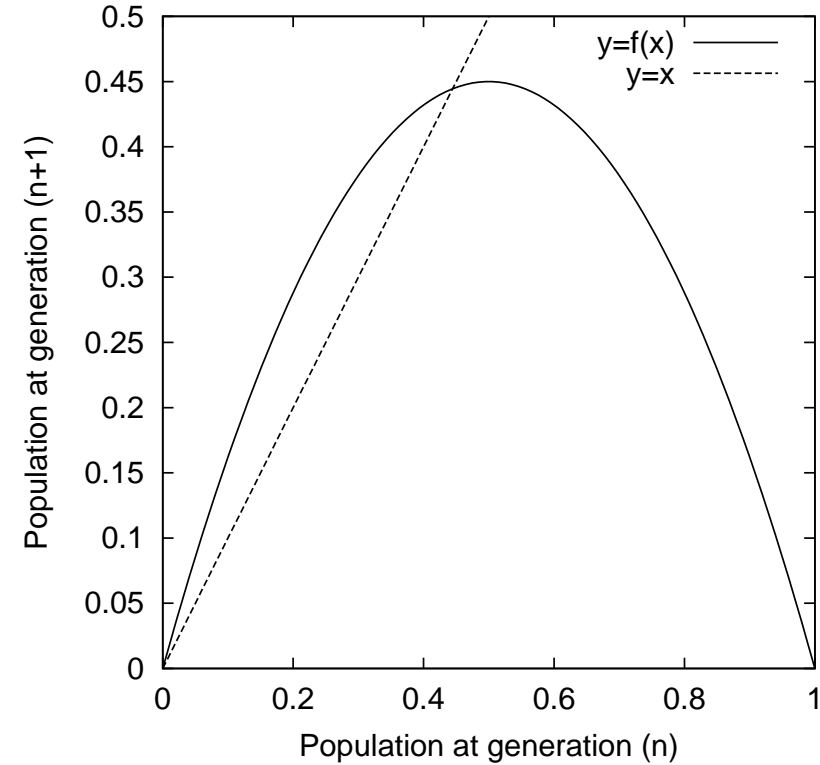


Figure 6.1: Population diagram for the discrete logistic model  $x_{n+1} = 1.8x_n(1 - x_n)$

## 6.2 Fixed Harvesting

Suppose that hunters are allowed to kill a fixed number  $H$  of the animal population. Then the unscaled population model is

$$N_{n+1} = N_n \left( r - \frac{N_n}{K} \right) - H. \quad (6.2)$$

In writing down equation (6.2) we are assuming that the animal species is culled after breeding, for otherwise, the culling would affect the number of births and deaths, and consequently the growth rate.

If we rescale this equation by defining  $x_n = \frac{N_n}{rK}$  we obtain

$$x_{n+1} = rx_n(1 - x_n) - h. \quad (6.3)$$

where  $h = \frac{H}{rK}$ . We shall assume that  $0 \leq h < 1$ .

**Question 6.3** *Why do we assume that  $0 \leq h < 1$ ?*

### Question 6.4

1. Show that when  $h = 0.066$  the fixed points of equation (6.3) are  $x_1^* = 0.1095$  and  $x_2^* = 0.3350$ .
2. Show that the associated eigenvalues are  $\lambda = 1.406$ , for the fixed point  $x_1^*$ , and  $\lambda = 0.594$ , for the fixed point  $x_2^*$ .
3. Using figure 6.2 (a) determine what happens to the population size  $x_n$  as  $n \rightarrow \infty$  if  $x_0 = \frac{0.8}{1.8}$ .

Biologically, the answer to this question means that if we cull  $h = 0.066$  of our population each year then the population will decrease from  $x^* = 0.444$  ( $0.8/1.8$ ) to  $x^* = 0.3350$ .

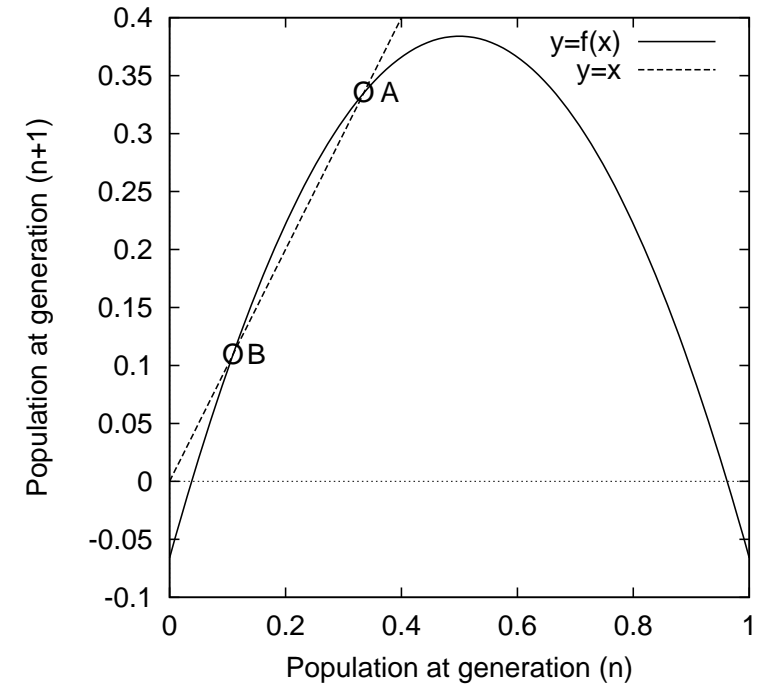


Figure 6.2: (a) Population diagram for the fixed harvest model  $x_{n+1} = 1.8x(1-x) - 0.066$ . There are two fixed points: an unstable fixed point  $x^* = 0.1095$  (B) and a stable fixed point  $x^* = 0.3350$  (A).

**Question 6.5** *What is the biological significance of choosing  $x_0 = \frac{0.8}{1.8}$  in question 6.4?*

**Question 6.6** *From the cobwebbing diagram describe what happens if the population is ever reduced below  $x^* = 0.1095$  (the population size at the unstable fixed point).*

**Question 6.7** *Suppose that at the end of the year the population size is  $x^* = 0.3350$  (the stable fixed point). Why might the population size in the next year reduce below  $x^* = 0.1095$  ?*

**Question 6.8** *Show that there are no fixed points when  $h = 0.12$ .*

Figure 6.2 (b) shows that the curves  $y = 1.8x(1 - x) - 0.12$  and  $y = x$  do not intersect over the range  $0 \leq x \leq 1$ .

Thus, there are no fixed points, as shown in your answer to question 6.8.

A cobweb diagram drawn on this figure shows that regardless of the initial size of the population the population size becomes negative, i.e. the animal species becomes extinct.

Thus if will cull  $h = 0.12$  the animal resource is not renewable.

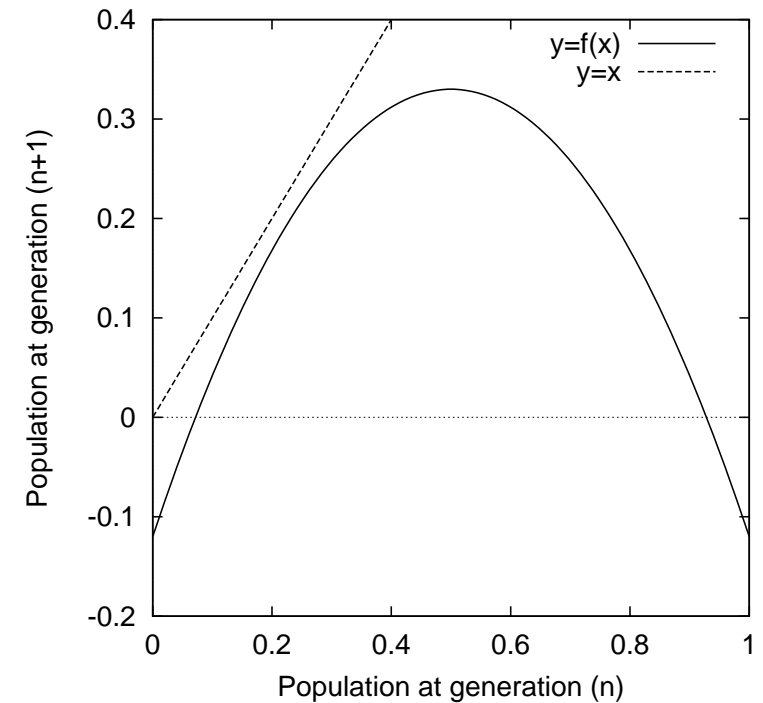


Figure 6.2: (b) Population diagrams for the fixed harvest model  $x_{n+1} = 1.8x(1 - x) - 0.12$ . There are no fixed points.

**Question 6.9** *Are there any circumstances in which it would be desirable to drive an animal population to extinction?*

We have seen that for two culling numbers, that is, for  $h = 0.066$  and  $h = 0.12$ , we had a different number of fixed points: 2 and 0, respectively. We would like to predict what happens for *any* fixed harvesting plan, that is, for any value of  $h$ .

The model equation is

$$x_{n+1} = 1.8x_n(1 - x_n) - h.$$

The fixed points are the solutions to the equation

$$x = \underline{1.8x(1 - x) - h},$$

where  $h$  is a constant and  $x$  is the as yet unknown fixed point. Multiplying out, this equation becomes

$$0 = \underline{0.8x - 1.8x^2 - h}.$$

Using the quadratic formula, the solutions are

$$x^* = \frac{0.8 \pm \sqrt{0.64 - 7.2h}}{3.6}. \quad (6.4)$$

There are three cases.

**Case one:**  $0.64 - 7.2h < 0$ . When  $h > 0.64/7.2$  there are no *fixed points*.

**Case two:**  $0.64 - 7.2h > 0$ . When  $h < 0.64/7.2$  there are two *fixed points*.

**Case three:**  $0.64 - 7.2h = 0$ . In this case there is only one *fixed point*

$$x^* = \frac{2}{9}.$$

Thus we have shown that when  $r = 1.8$  the critical value of the harvesting parameter is

$$h_c^r = 0.64/7.2 = \frac{4}{45}. \quad (6.5)$$

If  $h > h_{cr}$  then there are no fixed points and the population becomes extinct.

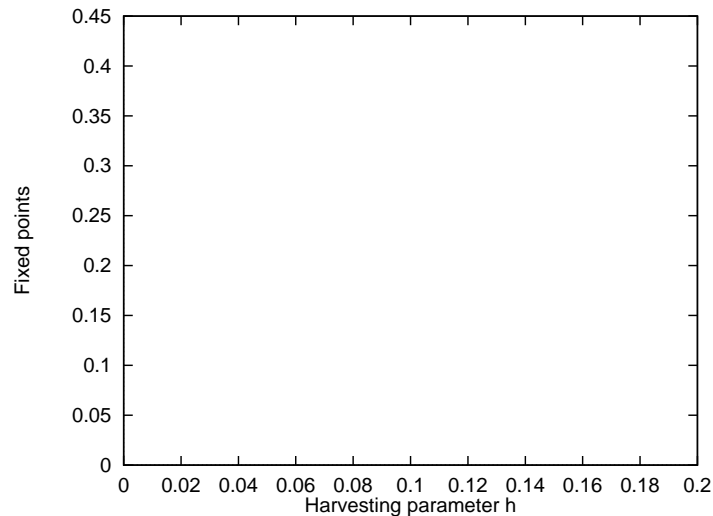


Figure 6.3: Variation of the fixed points ( $x^*$ ) in the fixed harvesting model  $x_{n+1} = 1.8x_n(1 - x_n) - h$  with the harvesting parameter  $h$ . If  $h > \frac{0.64}{7.2}$  there are no real fixed points.

- Figure 6.3 shows the variation of the fixed point  $x^*$  with the harvesting parameter  $h$ , as determined by equation (6.4).
- This figure shows that the maximum sustainable harvest is given by  $h_c^r = 4/45$ .
- However, the population of the harvested species is very fragile when  $h = h_{cr}$ .

If we were to cull too many animals then we would reduce the population size below  $x^* = 2/9$ . Should this happen, the population would become extinct unless we adjusted our hunting strategy until the population recovered.

Thus to be safe we should restrict  $h$  to some number slightly lower than the critical value. Even this would be risky, since the stable and unstable fixed points will be close together and a small catastrophe (such as slightly over-hunting) could result in the population dropping below the unstable fixed point, causing the population to start to die-out.

We have show that when  $r = 1.8$  the maximum sustainable harvest is  $h_c^r = \frac{4}{45}$ . How does this critical value depend upon  $r$ ?

**Question 6.10** Consider the logistic equation with fixed harvesting

$$x_{n+1} = rx_n(1 - x_n) - h,$$

where  $r$  ( $1 < r \leq 4$ ) and  $h$  ( $0 \leq h \leq 1$ ) are arbitrary. Show that harvesting is only sustainable if

$$h \leq \frac{(r-1)^2}{4r}.$$

*Hint.* Find the steady-states of equation (6.10). Hence show that there is a critical value of  $h$ ,  $h_c^r$ , above which there are no real fixed points.

### 6.3 Proportional harvest

Because of the unsatisfactory solution to the fixed harvesting model, we consider an alternative strategy, that is, to hunt or harvest, not a fixed number of the population, but a fixed proportion of the population. Let  $p$  represent the proportion of the population that is removed at the end of every time period. Then the unscaled population model is

$$N_{n+1} = N_n \left( r - \frac{N_n}{K} \right) - pN_n \quad 0 \leq p$$

and the scaled model ( $x_n = \frac{N_n}{rK}$ ) is

$$x_{n+1} = rx_n(1 - x_n) - px_n, \quad r = 1.8. \quad (6.6)$$

We are again assuming that the animal species is killed at the end of time period after breeding, the culling is assumed not to affect the number of births and deaths.

**Question 6.11** *What does the harvesting term  $-px_n$  ‘mean’?*

**Question 6.12** *If  $p$  is the proportional of the population that is killed, shouldn't the values of  $p$  be restricted to the range  $0 \leq p \leq 1$ ?*

### Question 6.13

1. *Show that the fixed points of equation (6.6) are  $x_1^* = 0$  and  $x_2^* = \frac{0.8-p}{1.8}$ .*
2. *Show that the eigenvalue of a fixed point  $x^*$  is given by*

$$\lambda = 1.8(1 - 2x^*) - p. \quad (6.7)$$

3. *Hence show that*
  - (a) *The trivial fixed point is unstable if either  $0 \leq p < 0.8$  or  $p > 2.8$  and stable if  $2.8 > p > 0.8$ .*
  - (b) *The non-trivial fixed point is stable if  $0 \leq p < 0.8$  and unstable if  $p > 0.8$ .*

Our analysis of this problem shows that the behaviour of the model when  $0 < p < 0.8$  is different from that when  $0.8 < p$ . Figure 6.4 shows population diagrams for the cases  $p = 0.6$  and  $p = 0.9$ .

**Question 6.14** Consider the proportiondla rvesting equation with  $r = 1.8$  and  $p = 0.9$

$$x_{n+1} = 1.8x_n(1 - x_n) - 0.9x_n$$

Suppose that  $x_0 = 0.6$ . Calculate  $x_1$ . Give both a biological interpretation of your result and a biological explanation of your result.

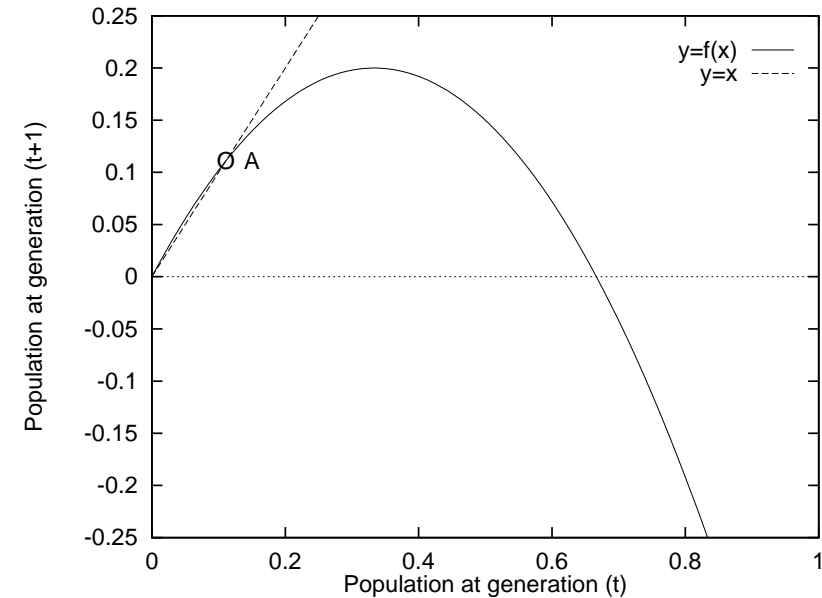


Figure 6.4: (a) Population diagrams for the proportional harvest model  $x_{n+1} = 1.8x_n(1 - x_n) - 0.6x_n$ . The origin is an unstable fixed point whilst the fixed point  $x^* = 0.111$  (A) is stable.

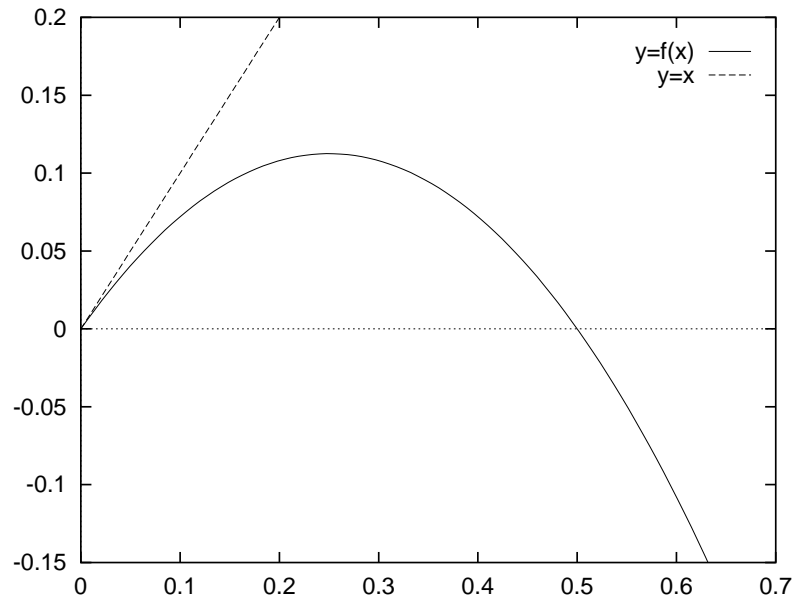


Figure 6.4: (b) Population diagrams for the proportional harvest model  $x_{n+1} = 1.8x_n(1 - x_n) - 0.9x_n$ . The origin is a stable fixed point and there is no positive fixed point.

The number of animals that are harvested ( $\mathcal{H}$ ) in generation  $n$  is

$$\mathcal{H} = px_n.$$

After sufficiently long time we hope that the population  $x_n$  will reach its static value  $x^*$ . When this happens the  $\mathcal{H}$  is given by

$$\mathcal{H} = px^* = \begin{cases} p \cdot \frac{0.8-p}{1.8} & 0 \leq p \leq 0.8 \\ 0 & 0.8 \leq p \leq 2.8 \end{cases} \quad (6.8)$$

We refer to equation (6.8) as the *yield equation* for proportional harvesting.

**Question 6.15** *Explain how the yield equation is derived.*

Figure 6.5 shows the graph of the harvested population versus the proportioned harvested.

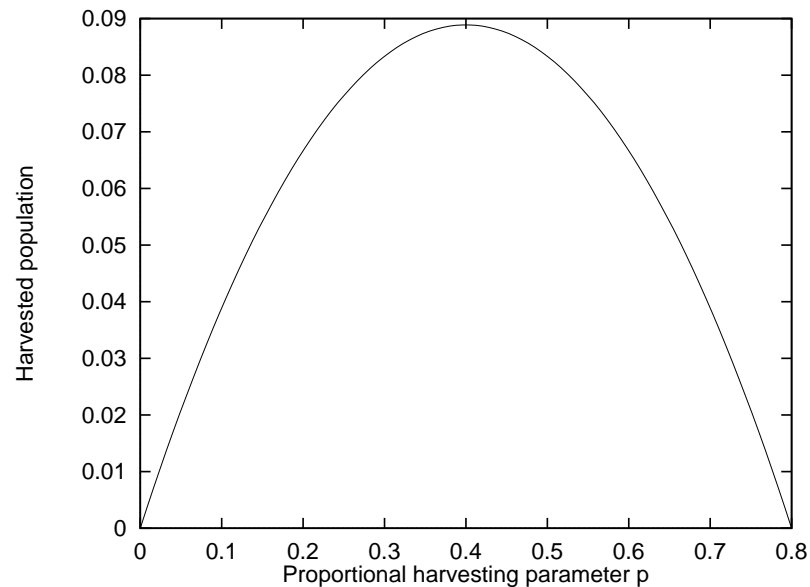


Figure 6.5: The dependence of the harvested population ( $\mathcal{H} = px^*$ ) with the proportional harvesting parameter ( $p$ ) in the proportional harvesting model  $x_{n+1} = 1.8x_n(1 - x_n) - px_n$ .

**Question 6.16** Show that the maximum harvest occurs at the point

$$(p, \mathcal{H}) = \left(0.4, \frac{0.16}{1.8}\right) = \left(0.4, \frac{4}{45}\right). \quad (6.9)$$

Compare this answer to your answer in the fixed harvesting model, equation (6.5).

- Thus the maximum harvest occurs when we harvest  $p = 0.4$  or 40% of the species each time period, and is  $\mathcal{H}_{max} = pu^* = \frac{0.16}{1.8} = \frac{0.64}{7.2}$ .
- Notice that the harvest is stable for any value of the proportional harvesting parameter in the range  $0 \leq p < 0.8$ .
- We have that when  $r = 1.8$  the maximum harvest occurs when  $p = 0.4$  and is  $\mathcal{H}_{max} = \frac{0.16}{1.8}$ . How do the optimal value of  $p$  and  $\mathcal{H}_{max}$  depend upon the value for  $r$ ?

**Question 6.17** Consider the logistic equation with proportional harvesting

$$x_{n+1} = rx_n(1 - x_n) - px_n, \quad (6.10)$$

where  $r(1 < r < 4)$  and  $p(0 \leq p \leq 1)$  are arbitrary.

1. Show that the fixed points of equation (6.10) are  $x^* = 0$  and  $x^* = \frac{r-1-p}{r}$ .
2. Show that the trivial fixed point is stable if

$$-1 < r - p < 1, \quad (6.11)$$

whilst the nontrivial fixed point is stable if

$$-1 < 2 + p - r < 1. \quad (6.12)$$

3. Show that the maximum harvest occurs when  $p = \frac{(r-1)}{2}$  and is  $\mathcal{H}_{max} = \frac{(r-1)^2}{4r}$ .
4. Compare the maximum harvest in the proportional harvesting model to that in the fixed harvest model, c.f. question 6.10.

**Question 6.18 (project)** *Suppose that in question 6.17 the value of  $r$  is such that when  $p = 0$  both fixed points are unstable. How does the behaviour of the model and the long-term average harvest depend upon  $p$ ? In particular, suppose that for a given value of  $r$  and  $p$  the solution is chaotic. Could it be that in the long-term we harvest more animals when the population is chaotic than when it is period-1?*

## 6.4 Comparison of harvesting strategies

We have investigated two harvesting strategies. In the first a fixed amount of the species is harvested in each time period. In the second a fixed proportion of the species is harvested in each time period. We have shown that the maximum sustainable harvest is the same for both policies.

When the proportional strategy is used the maximum harvest is stable and ecologically sound.

This is not the case when a fixed harvest is used: maximising the cull using a fixed harvest is a very risky strategy.

What happens in proportional harvesting is that, when  $p$  is slightly larger than the value at which the maximum harvest occurs, or when the population drops because of outside influences, the actual harvest  $\mathcal{H} = px_n$  decreases, which allows the population to recover.

In the fixed harvest recovery is not possible as we cull a fixed number of animals regardless of the current population size.

Therefore proportional harvesting is the superior harvesting strategy.

However, we should also recognise that it is harder to implement.

In practice we will not precisely know the current size of the population ( $x_n$ ) and so we can not give an exact value for our kill  $px_n$ . However, there are statistical methods for estimating the population size  $x_n$ .

- An important observation from the proportional harvesting model is that there is a value of  $p$  at which the maximum harvest occurs.
- In real life there is often a temptation for hunters/companies to increase their hunting efforts in order to increase the harvest.
- Here we have shown that, over the long term, this will reduce the population, and consequently the catch will drop.
- We are faced with the environmental reality that *if we increase our effort in harvesting, we will decrease our harvest.*

## 6.5 Advanced topics

When the ideas considered in this chapter are applied to real harvesting problems additional concerns have to be addressed. Two of these are:

**Population uncertainty:** In practice we do not know the exact size of the animal population, but we can estimate it using statistical techniques. Our model should include the statistical uncertainties in the value of  $x_n$ . How do these uncertainties change our recommendations?

**Economics:** We have asked the question “what is the maximum sustainable harvest?”

- In many applications a more pertinent question is “how do we maximise the *profit* from harvesting whilst maintaining a sustainable population”?
- We need to combine a biological model, for the population size, with an economics model, for the costs associated with harvesting.
- The mathematical problem is now to maximise the economic yield whilst maintaining a sustainable population.

The inclusion of these factors leads to interesting mathematical modelling problems.

## **6.6 Questions**

1. Do question 6.10
2. Do question 6.17

## **6.7 Revision of key ideas**

## **6.8 Concept map**

Draw a concept map for this chapter relating the aims/key ideas of the chapter. If you are unfamiliar with the idea of a concept map see appendix A.