2 First-Order Linear Difference Equations

2.1 Aims

In this chapter we focus on solving first-order difference equations of the form

$$x_n - a x_{n-1} = b(n)\ , \quad n \geq 1. \qquad (2.1)$$

In equation (2.1) a is a constant, $b(n)$ is a function of n and x_n is the state variable _____________ whose solution we seek to find.

Question 2.1 *Under what condition is equation (2.1) autonomous?*

After working through this chapter you should be able to:

1. solve first-order difference equations of the form:

$$x_n - a x_{n-1} = b(n)$$

2. check that a given expression is indeed the solution of the above equation.
3. set up and solve the difference equation relevant to a given scenario (ie. write down the word equation, define the variables and obtain the appropriate difference equation and solve to obtain the solution).

4. interpret your mathematical solution to gain insights into the original problem.

2.2 Carp Population

Example 2.1 (Ecology) *The population of carp in a lake increases through natural growth by 25% per year. Its size in the year $n = 0$ is 1100. If every year 300 carps are harvested, find an expression for the size of the population after year n. What is the population size in year $n = 2$? After how many years is the population size reduced below 500?*

We must first convert the **word** problem into a **difference equation**.

The key step is to first write down a word equation.

$$\left\{ \begin{array}{c} \text{change in} \\ \text{quantity} \end{array} \right\} = \left\{ \begin{array}{c} \text{reasons why the} \\ \text{quantity changed} \end{array} \right\}.$$

x_n = carp population size in year n.

$$\left\{ \begin{array}{c} \text{change in} \\ \text{population} \end{array} \right\} =$$

2.3 Solving the First-Order Difference Equation: The Case $b(n) = 0$

The difference equation

$$x_n - a x_{n-1} = 0\,, \quad n \geq 1 \tag{2.2}$$

has solution

$$x_n = x_0 a^n \tag{2.3}$$

where x_0 is the initial condition.

If $a > 1$ then we have exponential

______ whilst if $a < 1$ we have exponential _____.

Proof One: Given $x_n - ax_{n-1} = 0,\ n \geq 1$, we write down successive iterations, i.e.

$$\begin{aligned} x_1 &= ax_0 \\ x_2 &= ax_1 = \underline{\qquad\quad} = \underline{\qquad} \\ x_3 &= ax_2 = \underline{\qquad\quad\ } = \underline{\qquad} \\ \vdots\ &\vdots \qquad \vdots \qquad\quad \vdots \\ x_n &= ax_{n-1} = \underline{\qquad\qquad} = \underline{\qquad} \end{aligned}$$

$\square$

Proof Two: We can show that $x_n = a^n x_0$ is the solution of our difference equation by: (i) checking that it satisfies the initial equation and (ii) when it is substituted into the difference equation it satisfies the equation for all n.

When $n = 0$ the solution gives $x_0 = x_0$.

Substituting the expression into our difference equation, we obtain

$$\begin{aligned} a^n x_0 - a\left(a^{n-1} x_0\right) &= a^n x_0 - a^n x_0 \\ &= 0. \end{aligned}$$

Thus $x_n = a^n x_0$ is the solution of the given difference equation.

Example 2.2 *Find the solution of* $x_n = 1.5x_{n-1}$ *with* $x_0 = 2$.

Solution

2.4 Solving the First-Order Difference Equation: The Case $a = 1$

The difference equation

$$x_n - x_{n-1} = b(n)\,, \quad n \geq 1 \qquad (2.4)$$

has solution

$$x_n = x_0 + \sum_{p=1}^{n} b(p) \qquad (2.5)$$

where $x_0 =$ is the initial condition.

Here $b(n)$ is a function of n, possibly the constant function.

Again, we write out successive terms of the iteration process.

$$x_1 - x_0 = \underline{\hspace{4cm}}$$
$$x_2 - x_1 = \underline{\hspace{4cm}}$$
$$x_3 - x_2 = \underline{\hspace{4cm}}$$
$$\vdots \quad \vdots \quad = \vdots$$
$$x_{n-1} - x_{n-2} = \underline{\hspace{4cm}}$$
$$x_n - x_{n-1} = \underline{\hspace{4cm}}$$

Adding all these equations we obtain

$$\underline{\hspace{2cm}} = \underline{\hspace{5cm}}$$
$$= \sum_{p=1}^{n} b(p) \qquad \square$$

Question 2.2 *When $b(n) = 0$ equation (2.4) is a special case of equation (2.2), corresponding to the choice $a = 1$. Under these circumstances does equation (2.4) agree with equation (2.3)?*

Solution When $b(n) = 0$ the solution of equation (2.4) is given by $\underline{x_n = x_0}$. When $a = 1$ the solution to equation (2.2) is $\underline{x_n = x_0}$. Thus the two solutions agree.

Example 2.3 *Solve the difference equation* $x_n - x_{n-1} = 2n + 1$ *with* $x_1 = 2$. $(x_n = n^2 + 2n - 1)$

Hint: You will need to use appendix B.

Solution

2.5 Solving the First-Order Difference Equation: The General Case

The difference equation

$$x_n - ax_{n-1} = b(n) \tag{2.6}$$

has solution

$$x_n = x_0 a^n + \sum_{p=1}^{n} a^{n-p}\, b(p) \tag{2.7}$$

where x_0 is the initial condition.

The proof is not as straightforward as in the previous cases so I shall omit it.

Question 2.3 *Can you show that equation (2.7) is the solution of the general first-order difference equation (2.6)?*

Example 2.4 *Solve the linear difference equation*

$$x_n - 3x_{n-1} = 2\,, \quad x_1 = 2$$

Hint: You will need to use appendix C.
$(x_n = 3^n - 1)$

Solution

Question 2.4 *How would you check if this is the correct solution for example 2.4?*

Now we have the tools to solve our "carp problem" in its entirety.

Example 2.5 *The problem we were trying to solve was*

$$x_n - 1.25x_{n-1} = -300\,, \quad x_0 = 1100$$

Solution

Question 2.5

1. *How would your answer change if the culling rate was d instead of 300?*
2. *From your new equation, what is the maximum sustainable harvest of carp?*
3. *Explain why your answer to the last question can be obtained without solving the difference equation!*
4. *A fisheries manager asks you to recommend a value of d (the larger the value of d, the greater the profit). What value do you recommend? Justify your answer.*

2.6 Revision of key ideas

2.7 Concept map

Draw a concept map for this chapter relating the aims/key ideas of the chapter. If you are unfamiliar with the idea of a concept map see appendix A.

2.8 Questions

Question 1

Solve the following difference equations to obtain solutions in "closed form".

(a) $x_n - 2x_{n-1} = 0$

(b) $x_n = x_{n-1} + 3$

(c) $x_n + x_{n-1} = n$

(Hint: Arithmetic-Geometric Series $\sum_{k=1}^{n} (-1)^{n-k} k = \frac{1}{4}(2n+1) - \frac{1}{4}(-1)^n)$

Question 2

Find the solution of the following difference equation, simplifying as far as possible. Carefully explain each step of your solution.

$$x_n - x_{n-1} = n^2, \quad x_1 = 2, \quad n = 0, 1, \ldots$$

Question 3

A certain animal population has an initial size of 1000 and increases each year by 20% through normal birth and death processes. If each year 500 of the population are harvested and 400 are received into the population through immigration from neighbouring areas,

(a) write down, formally, the difference equation that describes the above scenario. (Make sure you define **all** variables and explain all terms.)

(b) How many years will it take for the population to exceed 2000 in size ?

Question 4. Consider the problem of modelling patient flow in a department of geriatric medicine. Each day the folowing activities occur:

- A number of new patients are admitted to the department for acute care.
- A fraction, α, of the current patients are treated and discharged.
- A fraction, β, of the current patients, unfortunately, die.
- A fraction of the current patients, γ, are transferred to another section.

(a) Write down a **word** equation that defines this problem.

(b) Write down, formally, the difference equation that describes the above scenario. Define **all** variables and explain your terms.

Question 5 *Carp problem revisited.* In lectures we derived the folowing difference equation to model carp numbers in a lake (example 2.1)

$$x_n = 1.25x_n - 300.$$

In obtaining this equation we have tacitly assume that harvesting only occurs *after* the carp have bred. What would be the corresponding equation if harvesting occurs *prior to* breeding? (Make sure you define **all** variables and explain all terms.)