8 Differential Equations: Introduction, Definitions and Basic Concepts

8.1 Aims

After working through this chapter you will be able to:

1. recognise DEs and have an appreciation of circumstances when they arise in physical problems;
2. understand what the following terms mean in relation to differential equations: initial conditions, order, solution;
3. see the link between the order of a differential equation and the number of initial conditions;
4. distinguish between linear and non-linear DEs *and* autonomous and non-autonomous systems;
5. verify if a given function is the solution to a specified differential equation;

8.2 Overview of Continuous Systems

In chapters 1–6 we used *difference equations* to model situations in which a physical setting can be reduced to a set of measurements made at a sequence of *equaly spaced* specified times.

In the remainder of this course we model **continuous systems**. Such systems are described by *differential equations*.

Differential equations are extensively used for mathematical modelling in areas such as economics & finance, engineering and science.

Problems that can be represented by differential equations include

- population dynamics
- a ball falling from a table
- a cup of hot coffee cooling
- the concentration of a chemical in a chemical reactor
- radioactive decay...etc.

8.3 What does $\frac{dy}{dt}$ mean?

8.3.1 Physical definition

The derivative $\frac{dy}{dt}$ represents the rate of change of the function $y(t)$ as t is varied.

- If $\frac{dy}{dt} > 0$ then the function $y(t)$ ________ as t is increased.
- If $\frac{dy}{dt} < 0$ then the function $y(t)$ ________ as t is increased.
- A point where $\frac{dy}{dt} = 0$ is known as a stationary point. At this point the tangent to the curve is flat.

Question 8.1 *Consider the function $y = f(t)$ shown in figure 8.1. For what values of $t \in [0, 2.5]$ is*

1. $\frac{dy}{dt} > 0$?
2. $\frac{dy}{dt} < 0$?
3. $\frac{dy}{dt} = 0$?

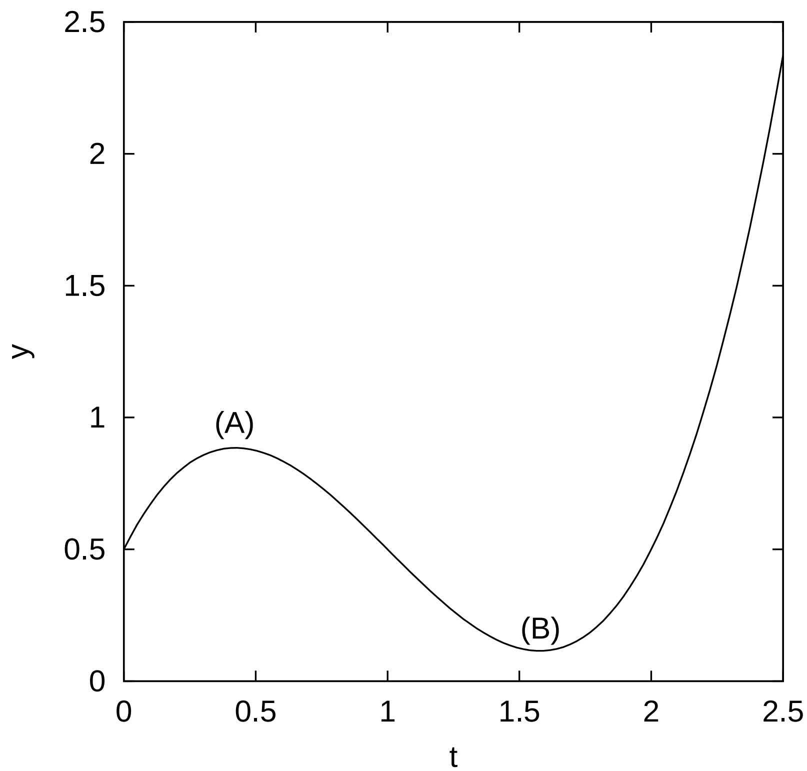

Figure 8.1: The function $y = f(t)$

8.3.2 Mathematical definition

The derivative of a function $f(x)$ with respect to the variable x is defined as

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

This represents the infinitesimal change in the function $f(x)$ with respect to the parameter x. The 'simple' derivative of a function f with respect to x may be denoted either as

$$\frac{\mathrm{d}f}{\mathrm{d}x}, \quad \text{or} \quad \underline{f'(x)} \quad \text{or} \quad \underline{f_x}.$$

When derivatives are taken with respect to time they are often denoted using the notation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \underline{\dot{x}}$$

When a derivative is taken n times, the notation

$$\frac{\mathrm{d}^n f}{\mathrm{d}x^n}$$

is used, with

$$\dot{x}, \ddot{x}, \dddot{x}$$

the corresponding notation when the derivative is respect to time.

8.4 Examples of Differential Equations

8.4.1 Population Biology

One important factor in modelling populations is whether the population grows continuously with time or in discrete jumps.

Many animal populations grow in discrete time, due to having well-defined breeding seasons.

Furthermore some insect populations have non-overlapping generations as the adults all die directly after giving birth.

However, some populations, such as humans (!), grow continuously in time.

In chapters 1–6 we considered discrete models for population biology. we now consider a continuous model.

Example 8.1 (Population model) *A population (X) changes size continuously due to births and deaths. Write down a differential equation to model the population size at time t $(X(t))$ assuming that*

- *there is a constant per-capita birth rate β;*
- *there is a constant per-capita death rate α;*
- *the change in population at any time t is proportional to the size of the population at time t.*

The processes represented in example 8.1 are sketched in figure 8.2.

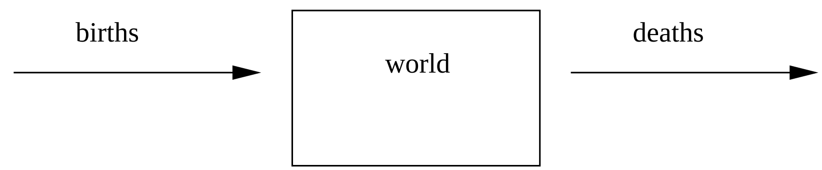

Figure 8.2: Population input-output diagram. From [Barnes & Fulford].

This sketch leads to a word equation describing a changing population

$$\left\{\begin{array}{c}\text{rate of}\\ \text{change of}\\ \text{population}\\ \text{size}\end{array}\right\} = \left\{\begin{array}{c}\text{rate}\\ \text{of}\\ \underline{\text{births}}\end{array}\right\} - \left\{\begin{array}{c}\text{rate}\\ \text{of}\\ \underline{\text{deaths}}\end{array}\right\} \tag{8.1}$$

Since the per-capita birth rate β is assumed constant, the overall birth rate at any time is the per-capita birth rate multiplied by the current population size.

Similarly, the overall death rate is the per-capita death rate multiplied by the current population size.

Thus we write

$$\left\{\begin{array}{c}\text{rate}\\ \text{of}\\ \text{births}\end{array}\right\} = \underline{\beta X(t)},$$

$$\left\{\begin{array}{c}\text{rate}\\ \text{of}\\ \text{deaths}\end{array}\right\} = \underline{\alpha X(t).} \tag{8.2}$$

Substituting (8.2) into (8.1) we obtain

$$\frac{\mathrm{d}X}{\mathrm{d}t} = \underline{\beta X - \alpha X}.$$

(Note that $X(t)$ can be writen as just X, since it is clear that X is evaluated at t).

Question 8.2 *Suppose that the species represented by X is humans. We have assumed that the birth-rate is proportional to the current population size. Do you have a comment on this?*

8.4.2 Lake pollution

Background Polution of lakes and rivers is a major problem. In order to reduce pollution an understanding of the processes involved is required.

A way of predicting how the situation may improve/decline as a result of changes in management practices is required.

To this end we need to be able to predict how the 'amount' of pollutant[a] varies over time and under different management strategies.

[a]If you have studied chemistry you will recognise that the variable is the *concentration* of pollutant.

Example 8.2 (Lake Pollution)

Consider a lake of volume V (m^3).

There is an input of polluted water from the river(s) flowing in to the lake, or due to a pollution dump into the lake, and an output of pollutant from the lake as water flows from the lake carrying some pollution with it.

Derive a differential equation for the concentration of pollutant (c) in the lake as a function of time.

State your assumptions.

The processes represented in this problem are sketched in figure 8.3.

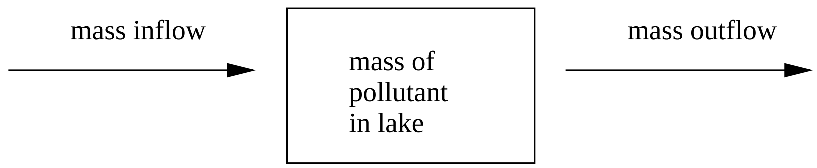

Figure 8.3: Input-output diagram for lake pollution. From [Barnes & Fulford].

This sketch leads to a word equation describing the mass of polution in the lake

$$\left\{\begin{array}{c}\text{rate of change}\\ \text{of mass of polutant}\\ \text{in lake}\end{array}\right\} = \left\{\begin{array}{c}\text{rate}\\ \text{mass of polutant}\\ \underline{\hspace{3cm}}\end{array}\right\} - \left\{\begin{array}{c}\text{rate}\\ \text{mass of polutant}\\ \underline{\hspace{3cm}}\end{array}\right\} \quad (8.3)$$

- We assume that the lake is continuously well mixed so that the pollutant is dispersed uniformly throughout the lake.
- Let $M(t)$ be the *mass* of polutant in the lake at time t. Then the concentration of pollutant in the lake at time t ($c(t)$) is given by

$$c(t) = \frac{M(t)}{\underline{V}}$$

- Let c_{in} be the *concentration* of pollutant flowing into the lake ($\mathrm{g\,m^{-3}}$).

- Let q be the rate at which water flows <u>out</u> of the lake in $\mathrm{m}^3/\mathrm{day}$. Since the volume of the lake is constant we have

$$\left\{\begin{array}{c}\text{flow}\\ \text{of water}\\ \underline{\text{into lake}}\end{array}\right\} =$$

$$\left\{\begin{array}{c}\text{flow}\\ \text{of water}\\ \underline{\text{out of lake}}\end{array}\right\} = \underline{q}.$$

Returning to equation (8.3) we have

$$\left\{\begin{array}{c}\text{rate of change}\\ \text{of mass of polutant}\\ \text{in lake}\end{array}\right\} =$$

$$\left\{\begin{array}{c}\text{rate}\\ \text{mass of polutant}\\ \text{enters lake}\end{array}\right\}$$

$$-\left\{\begin{array}{c}\text{rate}\\ \text{mass of pollutant}\\ \text{leaves lake}\end{array}\right\}$$

This translates into the differential equation for the changing *mass*

$$\frac{dM(t)}{dt} = \underline{qc_{\text{in}} - \frac{qM(t)}{V}}. \tag{8.4}$$

Now, since

$$M(t) = Vc(t)$$

we have that

$$\frac{dM(t)}{dt} = \underline{V\frac{dc(t)}{dt}}$$

(since the volume of the lake, V, is constant).

With this change of variable the differential equation for the mass (8.4) is transformed to a differential equation for the *concentration* of the pollutant in the lake,

$$V\frac{dc}{dt} = \underline{qc_{\text{in}} - qc}. \tag{8.5}$$

(Note that $c(t)$ can be writen as just c, since is it clear that c is evaluated at t).

Question 8.3 *Check that the units in equation* (8.5) *are constant.*

8.5 Formal Definitions

Definition 8.1 ***(Differential Equation)*** *A differential equation (frequently abbreviated DE) is an equation involving an unknown function and certain derivatives ________ of this function.*

Definition 8.2 ***(Ordinary Differential Equation)*** *An ordinary differential equation (frequently abbreviated ODE) is an equation involving "ordinary" derivative of the unknown function. For example,*

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{2x} + \sin x.$$

Definition 8.3 ***(Independent and dependent variables)*** *In the derivative $\frac{dy}{dx}$ the function that is being differentiated, in this case y, is the dependent variable whilst the parameter that the differentiation is occurring with respect to, in this case x, is the independent variable.*

So in example 8.1 the dependent variable is the size of the population whilst the independent variable is time.

Question 8.4 *Why is the independent variable called the independent variable? Why is the dependent variable called the dependent variable?*

Definition 8.4 ***(Order of a differential equation)*** *The order of a differential equation is the greatest ______ number of derivatives in any term of the differential equation. For example, the ordinary differential equation*

$$\frac{d^4u}{dx^4} = u^5$$

is of ______ order whilst the ordinary differential equation

$$\frac{dv}{dy} + y^2v^3 + v = 3$$

is of ___ order.

Example 8.3 *What is the order of the following differential equations. Justify your answer.*

(a) $\dfrac{d^3y}{dx^3} + 2y = 7$

(b) $\dfrac{dy}{dx} + 1 = x^2$

(c) $\left(\dfrac{dy}{dx}\right)^2 + \dfrac{dy}{dx} + 4y = 7$

(d) $\left(\dfrac{d^2x}{dt^2}\right)^4 + \dfrac{1}{t}\dfrac{dx}{dt} + 7x = e^{4t}$

Solution

Definition 8.5 ***(Linear and non-linear difference equations)*** *A differential equation is said to be linear if the dependent variable and* ***all*** *of its derivatives appear linearly. For example, the equation*

$$x^3 \frac{d^3y}{dx^3} + \frac{dy}{dx} + \cos x = 0$$

is a linear equation, whilst the equation

$$y \frac{dy}{dx} = 1$$

is termed ***non-linear****.*

A linear differential equation of order n can be written in the form

$$a_n(t) \frac{\mathrm{d}^n y}{\mathrm{d}t^n} + a_{n-1}(t) \frac{\mathrm{d}^{n-1} y}{\mathrm{d}t^{n-1}} + \cdots + a_1(t) \frac{\mathrm{d}y}{\mathrm{d}t} + a_0(t)\, y = Q(t).$$

Example 8.4 *State whether the following differential equations are linear or non-linear. Justify your answer.*

(a) $\frac{dy}{dx} = 2y$.

(b) $\frac{d^2y}{dx^2} + 3\sin y = 0$.

(c) $y'' + 4y' + 3y = \sin x$.

(d) $y'' + yy' + 3y = \sin x$.

(e) $x^2y'' + xy' + x^2y = 8$.

(f) $(y'')^2 + y' + y = x^2$

Definition 8.6 (Initial conditions) *The initial condition(s) are the condition(s) at the initial time $t = t_0$ from which the solution to a differential equation evolves. A system with initial conditions specified is known as an initial value problem (commonly abbreviated IVP).*

For a first-order differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(y, t)$$

the initial condition species the value of the dependent variable, here y, at an initial value of the independent variable, here t. This is written

$$y(t = t_0) = y_0,$$

or more succinctly

$$y(t_0) = y_0.$$

For instance, the initial condition

$$y(0) = 1,$$

means that when $t =$ _ the value of y is _

Definition 8.7 ***(Order and initial conditions)*** *To solve a equation of order m that is an initial value problem there must be m initial conditions.*

For a mth order differential equation the initial conditions are the value of the dependent variable at a specified value of the independent variable and the values of the first $m-1$ derivatives.

For example, to solve the second-order equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + x\frac{\mathrm{d}y}{\mathrm{d}t} = x^2$$

the values $y(t_0)$ and $\left.\frac{\mathrm{d}y}{\mathrm{d}t}\right|_{t=t_0}$ need to be specified.

Question 8.5 *Consider the differential equation*

$$\frac{d^3 y}{dt^3} - \frac{d^2 y}{dt^2} - \frac{dy}{dt} = 0$$

1. *What is the order of this equation?*
2. *Is it a linear or non-linear differential equation?*
3. *How many initial conditions do we need to solve the initial value problem?*
4. *What values must be specified before we can solve the initial value problem?*

Definition 8.8 (Solution) *A solution of a differential equation is a function of the independent variable which when substituted into the differential equation satisfies the governing equation. Furthermore, if provided, it satisfies ________ the initial condition(s).*

Example 8.5 *Show that the function $y = xe^x$ is a solution of the differential equation*

$$y'' - 2y' + y = 0.$$

Solution

To show this we compute

$y' =$

____

$y'' =$

____

Substituting these into the ODE, we obtain

$$y'' - 2y' + y =$$

$$= 0.$$

Hence $y = xe^x$ is a solution of the ODE. □

All we are doing here is to substitute the form of the solution into our differential equation to verify that the LHS=RHS.

Example 8.6 *Verify that*

$$p = \frac{k}{r} + \left[p_0 - \frac{k}{r}\right] e^{rt}$$

is the solution of the DE

$$\frac{dp}{dt} = rp - k, \quad p(0) = p_0.$$

Solution

Hence $p = \frac{k}{r} + \left[p_0 - \frac{k}{r}\right] e^{rt}$ is the solution of the given differential equation. □

Definition 8.9 ***(Autonomous and non-autonomous)*** *An ordinary differential equation is autonomous if the independent variable does not appear explicitly in the equation. For example,* $y_{xxx} + (y_x)^2 = y$ *is autonomous whilst* $y_x = x$ *is not.*

Question 8.6 *Which of the following differential equations are autonomous? (Give reasons.)*

(a) $\frac{d^2y}{dx^2} = \frac{dy}{dx} + y + x^2$

(b) $\frac{dx}{dt} - tx = 1$

(c) $\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{1}{y}$

8.6 Some comments on modelling

- The most challenging aspect in using mathematical modelling to study problems is translating the description of the 'physical' situation into mathematical equations.
- Although real-world systems can be complicated, we should aim to simplify the problem by making assumptions which seem reasonable. Our aim is to isolate the essential features of the problem and to eliminate the non-essential.

- Once we have developed the model, we should compare the predictions gained from the model with data from the system.
- If the model and the system agree, we can be confident that the assumptions made in creating the model were reasonable, and can then use the model to make predictions.
- If the model and the system disagree, then we must study the system more carefuly, make our assumptions more realistic and adjust the model accordingly.
- In either case we learn more about the system by comparing it to the model.

8.7 Revision of key ideas

8.8 Concept map

Draw a concept map for this chapter relating the aims/key ideas of the chapter. If you are unfamiliar with the idea of a concept map see appendix A.