13 Harvesting

13.1 Introduction

In this chapter we study the effect on a population of the removal of members of the population at a specified *rate*. If a population modelled by a DE

$$x' = f(x)$$

is subject to a harvest at a *rate* $h(t)$ members per unit time for some given function $h(t)$ then the harvested population is modelled by the differential equation

$$x' = \underline{f(x) - h(t)}.$$

Question 13.1 *Suppose that a graph of the function $y = f(x)$ is given. Explain how the graph of the function $y = f(x) - H$, $H > 0$ can be obtained without calculating values of the function.*

13.2 Aims

During this chapter you will:

1. Analyse simple models for the management of harvested species.
   - Learn what is meant by a *constant yield* harvesting strategy.
   - Learn what is meant by a *constant effort* harvesting strategy.
2. Relate *mathematical* results to the *physical problem*.
   - Understand how constant yield harvesting can drive a population to extinction.
   - Understand how constant effort harvesting can drive a population to extinction.
   - Learn how to optimise the yield from both the constant yield and constant effort harvesting strategies.
   - Understand why constant effort harvesting is a beter strategy to manage a sustainable resource than constant yield.
   - To appreciate why in practice constant yield harvesting might be preferred to constant effort harvesting.

13.3 The logistic differential equation: A quick recap

In this chapter we will assume that the population model in the absence of harvesting is the logistic DE. Recall that the logistic DE is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = rx\left(1 - \frac{x}{K}\right), \quad x(t=0) = x_0. \tag{13.1}$$

Before investigating how harvesting effects the dynamics of our model we should first determine what happens when there is no harvesting.

Question 13.2

1. *Show that the steady-state solutions of equation (13.1) are $x^* = 0$ and $x^* = K$.*
2. *Show that the eigenvalues are $\lambda = r$ when $x^* = 0$ and $\lambda = -r$ when $x^* = K$.*
3. *Using figure 13.1 determine what happens to the population size $x(t)$ as $t \to \infty$ for any, strictly positive, initial condition. How does the evolution of the population size depend upon the choice of initial condition?*

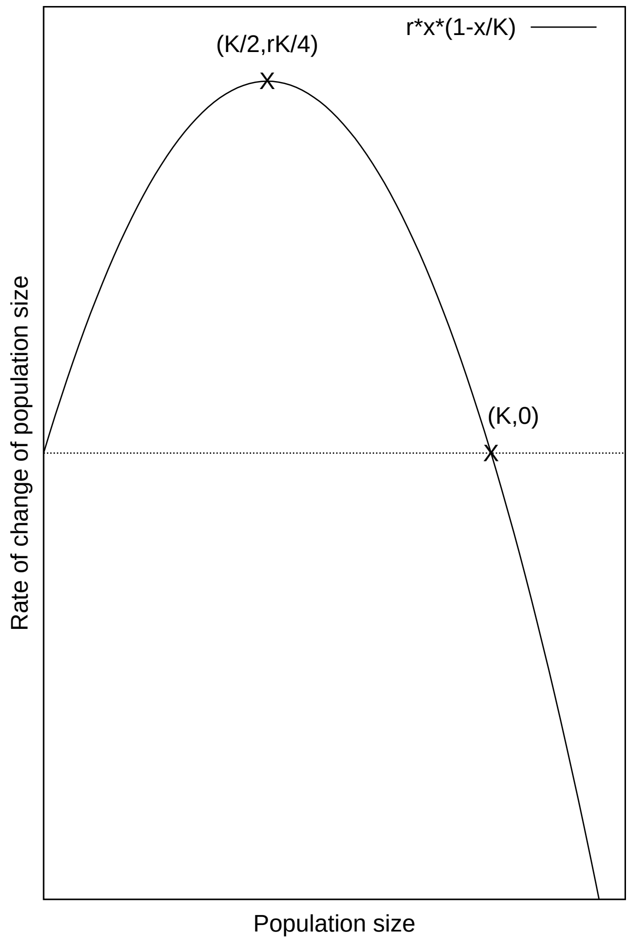


Figure 13.1: Sketch of the function $y = rx(1 - x/K)$.

13.4 Constant yield harvesting

If the function $h(t)$ is a <u>constant</u>, <u>H</u>, so that members are removed at the constant *rate* of <u>H</u>, per unit time the model is

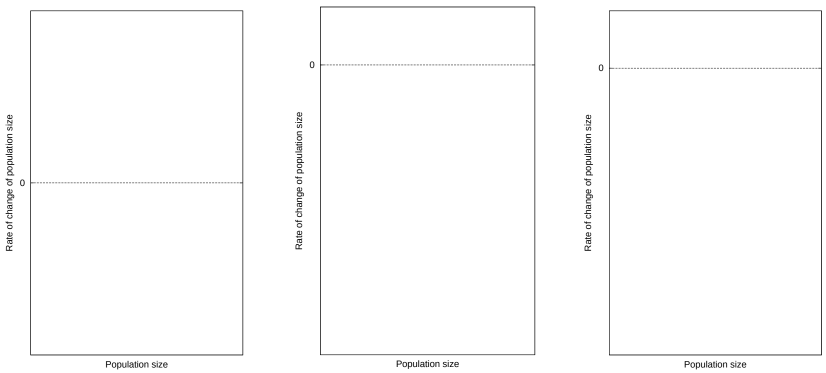
$$\frac{\mathrm{d}x}{\mathrm{d}t} = \underline{rx\left(1 - \frac{x}{K}\right) - H}, \; x(t=0) = x_0. \tag{13.2}$$

This type of harvesting is called <u>*constant rate*</u> or <u>*constant yield*</u> harvesting. It arises when a <u>*quota*</u> is *specified* (for example, through permits as in deer-hunting seasons in many parts of the USA or by agreement as sometimes occurs in whaling).

Question 13.3 *By considering how the graph of the function* $f(x) = rx\left(1 - \frac{x}{K}\right) - H$ *changes as* H *increases from zero show graphically that there is a critical value of* H, H_{cr}, *such that harvesting is sustainable if* $H < H_{cr}$ *and unsustainable if* $H > H_{cr}$.

1. *Sketch the function* $f(x) = rx\left(1 - \frac{x}{K}\right) - H$ *in figure 13.2*
   (a) *for a value of* H *with* $H < H_{cr}$,
   (b) *for* $H = H_{cr}$,
   (c) *for a value of* H *with* $H > H_{cr}$.

2. *For each of the figures you have drawn identify the steady-state solutions of the associated differential equation (should they exist) and determine their stability. Explain how the evolution of the population size depend upon the choice of initial condition, commenting on the implications of your answer for management of the resource as appropriate.*



(a) $H < H_{\rm cr}$ (b) $H = H_{\rm cr}$ (c) $H > H_{\rm cr}$

Figure 13.2: The three types of behaviour in the logistic differential equation with constant yield harvesting.

Note

1. For $H > 0$ $x = 0$ is <u>not</u> a steady-state solution of equation (13.1). However, once the solution has reached $x = 0$ the species has become extinct ______.

2. For $H < H_{\rm cr}$ there are <u>*two*</u> steady-state solutions. At the critical value, $H_{\rm cr}$, the two equilibria coalesce and annihilate each other — when $H > H_{\rm cr}$ there are <u>*no*</u> steady-state solutions.

We now determine the critical value of the harvesting parameter H, $H_{\rm cr}$.

Question 13.4 *Consider the logistic differential equation with constant yield harvesting*

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right) - H, \quad x\left(t = 0\right) = x_0.$$

1. *Show that the steady-state solutions of this model are given by*

$$x^*_{\pm} = \frac{rK \pm \sqrt{rK\left(rK - 4H\right)}}{2r}.$$

2. *Hence show that there is a critical value of H, H_{cr}, such that harvesting is only sustainable if $H \leq H_{cr}$. Determine the value H_{cr}.*

3. *Using your answers to question 13.3 determine the stability of the steady-state solutions x_- and x_+.*

4. *Use your answers to the previous parts of this question to draw a steady-state diagram for this model in figure 13.3.* ***Hint***. *Reread chapter (12.5) to remind yourself what a steady-state diagram is.*

5. *Would you recommend harvesting with $H = H_{cr}$? Justify your answer.*

Harvesting

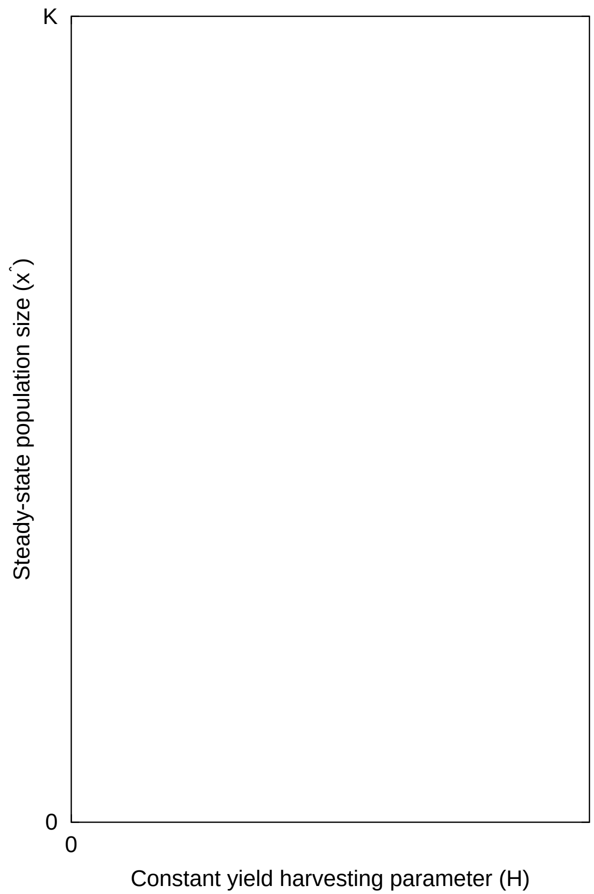


Figure 13.3: Schematic steady-state diagram for the logistic differential equation with constant yield harvesting.

13.5 Constant effort harvesting

If the function $h(t)$ is a *linear* function of pop^{n} size, $\underline{h(t) = Ex(t)}$, the model is

$$x' = \underline{f(x) - Ex}.$$

This type of harvesting is called *proportional* or *constant effort harvesting*. It arises in the modelling of fisheries, where it is often assumed that the number of fish caught per unit time is proportional to the effort expended in fishing (E). This fishing effort may be measured, for example, by the number of boats fishing at a given time. This is a reasonable hypothesis for many actual fisheries.

Question 13.5 *Can the assumption that the catch is proportional to effort be questioned?*

Answer Yes. More effort per fish caught may be necessary if the fish population is very small. Additionaly, when the effort is very high the total catch should satuate.

If the population is governed by a logistic model then the harvested model is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = rx\left(1 - \frac{x}{K}\right) - Ex, \quad r > 0,\ E \geq 0. \tag{13.3}$$

Question 13.6

1. *Show that the steady-state solutions of equation* (13.3) *are* $x_1^* = 0$ *and* $x_2^* = \frac{K(r-E)}{r}$.
2. *Show that the eigenvalue of a steady-state solution* x^* *is given by*

$$\lambda = r\left(1 - \frac{2x^*}{K}\right) - E. \tag{13.4}$$

3. *Hence show that*
   (a) *The trivial steady-state solution is unstable if* $r > E$ *and stable if* $r < E$.
   (b) *The non-trivial fixed point is stable if* $E < r$ *and unstable if* $E > r$.
4. *Use your answers to the previous parts of this diagram to draw a steady-state diagram for this model in figure 13.4.* ***Hint****. Reread chapter* (12.5) *to remind yourself what a steady-state diagram is.*

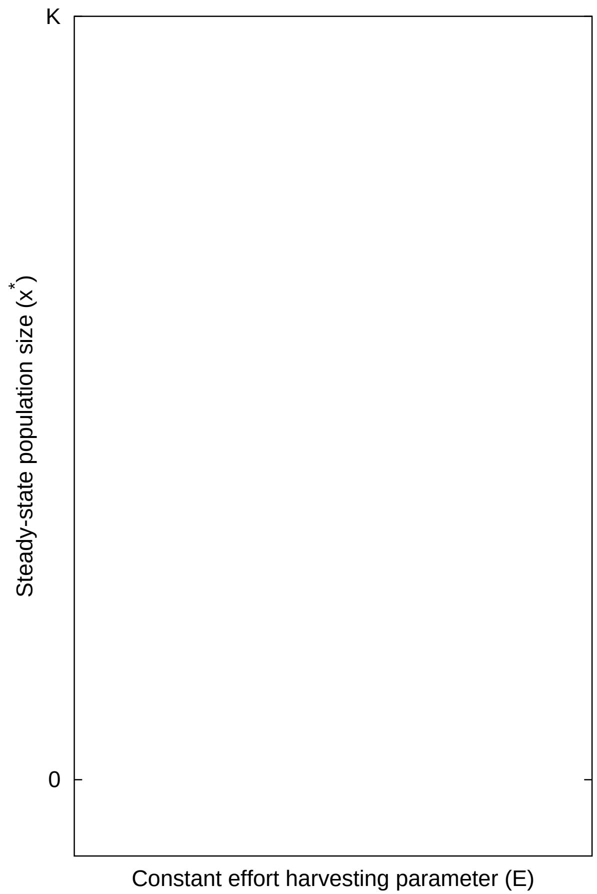


Figure 13.4: Schematic steady-state diagram for the logistic differential equation with constant effort harvesting.

Our analysis of this problem shows that the behaviour of the model when $E < r$ is different from that when $r < E$.

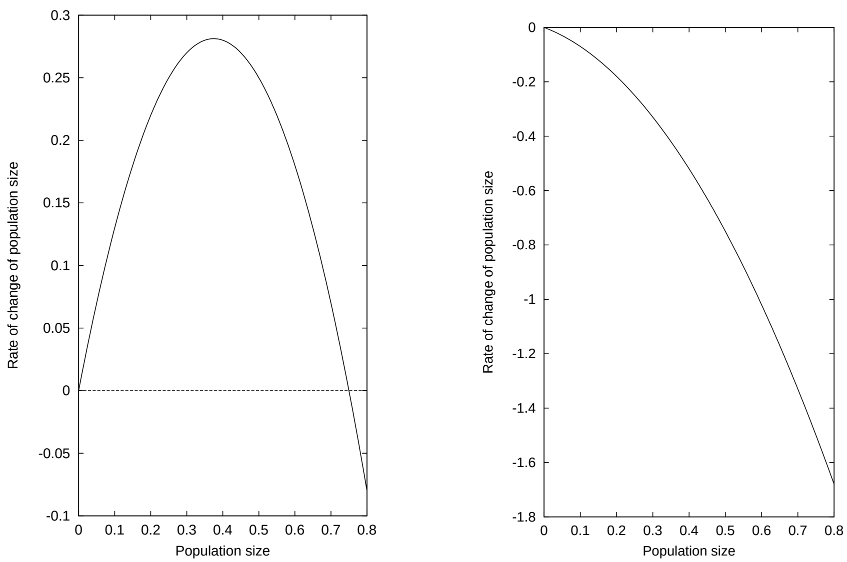
Question 13.7 *Using figure 13.5 determine what happens to the population size $x(t)$ as $t \to \infty$ for any, strictly positive, initial condition for the model*

$$\frac{dx}{dt} = 2x(1-x) - Ex, \quad x(0) = x_0$$

with

1. $E = 0.5$,
2. $E = 2.5$,

commenting on the implications of your answers for management of the resource as appropriate.



(a) Proportional harvesting parameter $E = 0.5$.

(b) Proportional harvesting parameter $E = 2.5$.

Figure 13.5: Population diagrams for the proportional harvest model $\frac{dx}{dt} = 2x(1-x) - Ex$.

For a given value of the effort E the long-term yield ($\mathcal{Y}$) is

$$\mathcal{Y} = \underline{Ex^*(E)}, \qquad (13.5)$$

where $\underline{x^*(E)}$ is the *stable* steady-state of the model. Equation (13.5) is the *yield equation* for proportional harvesting. (Note that although $\mathcal{Y}$ is known as the yield, it is actually the rate of harvesting under steady-state conditions).

For equation (13.3) the yield equation is

$$\mathcal{Y} = Ex^* = \begin{cases} \dfrac{E \cdot \frac{K(r-E)}{r}}{\underline{0}} & E < r \\ & E > r \end{cases} \quad (13.6)$$

Question 13.8 *Derive the yield equation, given above, for equation* (13.3).

Figure 13.6 shows the *yield effort curve*, the graph of yield against effort, for equation (13.3). Note that the yield increases to a maximum, called the *maximum sustainable yield (MSY)*, and then decreases to zero.

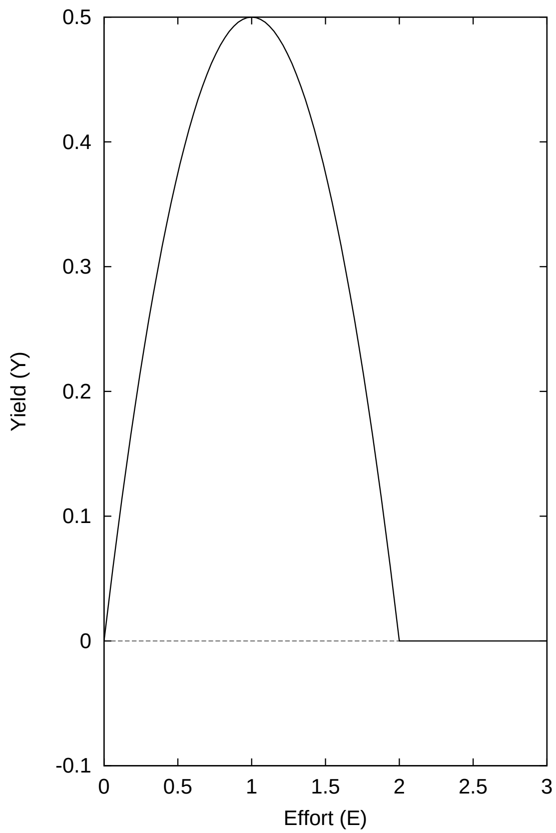


Figure 13.6: The *yield effort curve* for the proportional harvesting model $\frac{dx}{dt} = 2x(1-x) - Ex$.

Question 13.9 *Show that the maximum yield occurs at the point*

$$(E, \mathcal{Y}) = \left(\frac{r}{2}, \frac{rK}{4}\right) \qquad (13.7)$$

Compare this answer to your answer in the fixed harvesting model, question 13.4.

Thus the maximum harvest occurs when the effort is $E = \frac{r}{2}$ and is $\mathcal{Y}_{\max} = Ex^* = \frac{rK}{4}$. Notice that the harvest is stable for any value of the proportional harvesting parameter in the range $E < r$ and that it is counter-productive in the long-term to increase the effort beyond $\frac{r}{2}$ in that it decreases the yield.

13.6 Comparison of harvesting strategies

We have shown that the maximum sustainable harvest is the *same* for both the fixed and proportional harvesting policies. When the proportional strategy is used the maximum harvest is stable and ecologicaly sound. However, this is not the case when a fixed harvest is used: maximising the cull using a fixed harvest is a very risky strategy.

Thus our conclusions about the two harvesting strategies are the same as when we compared them within the context of a difference equation model.

13.7 Revision of key ideas

13.8 Concept map

Draw a concept map for this chapter relating the aims/key ideas of the chapter. If you are unfamiliar with the idea of a concept map see appendix A.