

## 9 First-Order Differential Equations

### 9.1 Aims

In this chapter we focus on solving first-order differential equations. We can write a first-order differential equation in the general form

$$\frac{dy}{dx} = f(y, x).$$

We will consider three cases, corresponding to special cases of the function  $f(y, x)$ . These cases are:

**The integrable case:** The function  $f(y, x)$  is independent of  $y$ , i.e.

$$f(y, x) = g(x).$$

**The separable case:** The function  $f(y, x)$  can be written in the form

$$f(y, x) = \frac{g(x)}{h(y)}$$

**The integrating factor case:** This applies when the function  $f(y, x)$  can be written in the form

$$f(y, x) = -P(x)y + g(x).$$

After working through this chapter you should be able to:

1. solve first-order DEs that are:
  - integrable;
  - separable.
2. solve first-order DEs using the integrating factor method.
3. check that a given expression is indeed the solution of a given differential equation.

## 9.2 Solving a first-order differential equation: Integrable equations

The first-order differential equation

$$\frac{dy}{dx} = g(x)$$

has solution

$$y = \int g(x) dx + c.$$

where  $c$  is an arbitrary constant.

Let us solve a differential equation of this type

**Example 9.1** Find the solution of

$$\frac{dy}{dx} - \sin x + x^2 = 0$$

### Solution

To solve  $\frac{dy}{dx} - \sin x + x^2 = 0$  for  $y(x)$   
write

$$\begin{aligned} \frac{dy}{dx} &= \sin x - x^2 \\ \Rightarrow y(x) &= \int (\sin x - x^2) dx \\ &= \underline{\hspace{2cm}} \end{aligned}$$

**Question 9.1** Check that

$y(x) = -\cos x - \frac{1}{3}x^3 + c$  is the solution  
to example 9.1.

$$\left(\frac{dy}{dx} - \sin x + x^2 = 0\right)$$

**Example 9.2** Find the solution of

$$\frac{dy}{dx} = ax.$$

**Solution**

$$y(x) = \underline{\hspace{2cm}}$$

$$= \underline{\hspace{2cm}}$$

where  $c$  is an arbitrary constant.

### 9.3 Solving a first-order differential equation: Separable equations

If

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}$$

then

$$\int h(y) dy = \int g(x) dy.$$

Remember the constant of integration  $+c$ , when integrating.

**Example 9.3** Solve  $\frac{dy}{dx} = \frac{y}{1+x}$ .

**Solution**

$$\int \frac{1}{y} dy = \int \frac{1}{1+x} dx$$

$$\Rightarrow \ln |y| =$$

$$\Rightarrow y =$$

$$=$$

$$= k(1+x)$$

where  $k = e^c$  is a constant.

**Example 9.4** Solve  $\frac{dy}{dx} = \frac{-x}{y}$  with the initial condition  $y(1) = 1$ .

**Solution**

$$\int y dy = - \int x dx$$

$$\Rightarrow \frac{1}{2} y^2 =$$

where  $c$  is an arbitrary constant. Applying the initial condition  $y(1) = 1$  implies  $c = \underline{\quad}$ , giving the solution

$$\frac{1}{2} y^2 =$$

$$y^2 + x^2 = \underline{\quad}$$

**Question 9.2** What object does the solution to example 9.4 describe?

**Example 9.5** Solve  $\frac{dy}{dt} = -ay$  with the initial condition  $y(0) = y_0 > 0$ .

**Solution**

$$\int \frac{1}{y} dy = -a \int dt$$

giving the solution

$$y = y_0 e^{-at}.$$

**Example 9.6** Solve  $V \frac{dy}{dt} = q(y_{in} - y)$  with the initial condition  $y(0) = y_0 > 0$  subject to  $q > 0$  and  $y_{in} > 0$ .

**Solution**

$$\int \frac{1}{y_{in} - y} dy = -\frac{q}{V} \int dt$$

with solution  $y = y_{in} - (y_{in} - y_0) e^{-qt/V}$ .

## 9.4 Solving a first-order linear differential equations: Integrating factors

To solve the general first-order linear differential equation

$$\frac{dy}{dx} + P(x)y = g(x)$$

calculate the integrating factor

$$R(x) = \exp \left[ \int P(x) dx \right]$$

then multiplying both sides of the equation by the integrating factor gives

$$R(x) \frac{dy}{dx} + R(x)P(x)y = R(x)g(x) \quad (9.1)$$

$$\frac{d}{dx}(R(x)y) = R(x)g(x) \quad (9.2)$$

$$\Rightarrow y = \frac{1}{R(x)} \left( \int R(x)g(x) dx + c \right)$$

where  $c$  is an arbitrary constant.

**Question 9.3** Show that the LHS of equation (9.2) is equivalent to the LHS of equation (9.1).

**Example 9.7** Solve the first-order linear differential equation

$$x \frac{dy}{dx} + 4y = x^3 - x, \quad \text{for } x > 0.$$

**Solution** The equation is divided through by  $x$  to give

$$\frac{dy}{dx} + \frac{4}{x}y = x^2 - 1.$$

1. Determine the integrating factor

Multiply both sides of the differential equation by the integrating factor ( $x^4$ ) and integrate.

$$\frac{d}{dx} (x^4 y) = x^4 (x^2 - 1)$$

**Question 9.4** *Check that*

$y = \frac{x^3}{7} - \frac{x}{5} + \frac{c}{x^4}$  *is the solution for example 9.7.*

$$\left( \frac{dy}{dx} + \frac{4}{x}y = x^2 - x. \right)$$

## 9.5 Revision of key ideas

## 9.6 Concept map

Draw a concept map for this chapter relating the aims/key ideas of the chapter. If you are unfamiliar with the idea of a concept map see appendix A.