10 Applications of First-Order Differential Equations: Lake Pollution

10.1 Aims

In this chapter we apply the techniques that we learnt in chapter 9 to a variety of problems that are based upon the model developed in chapter 8.4.2 for lake pollution.

After working through this chapter you should be able to

1. Find the solution of some differential equations modelling lake pollution.
2. Use your solution to answer problems on lake pollution.
3. Explain the physical meaning of terms in more complicated models of lake pollution.

10.2 The lake polution model

In section 8.4.2 we derived the following model for the rate of change of pollutant *concentration* with time in a lake

$$V\frac{\mathrm{d}c}{\mathrm{d}t} = qc_{\mathrm{in}} - qc, \quad c\,(t=0) = c_0. \quad (10.1)$$

In equation 10.1:

V is the volume of the lake (measured in units of ___),

c is the concentration of pollutant in the lake (measured in units of $\mathrm{g\,m^{-3}}$),

t is time (measured in units of day),

q is the flowrate of water through the lake (measured in units of _______),

$$\boxed{V\frac{\mathrm{d}c}{\mathrm{d}t} = qc_{\mathrm{in}} - qc, \quad c\,(t=0) = c_0.}$$

c_{in} is the concentration of pollutant in the water flowing into the lake (measured in units of $\mathrm{g\,m^{-3}}$) and

c_0 is the concentration of pollutant in the lake at at time $t = 0$ (measured in units of $\mathrm{g\,m^{-3}}$).

Remember that concentration is mass divided by volume.

10.2.1 Polutant dumped into a clean lake: Fixed flow-rate

Example 10.1 *(**[Barnes & Fulford], chapter 2.5**) At time $t = 0$ polutant is dumped into a clean lake giving an initial concentration of polutant c_0. Suppose that only fresh water flows into the lake. How long will it take for the lake's polution level to reach 5% of its initial value?*

Solution From the question we have $c_{\text{in}} = 0$. Thus the model is

$$V\frac{\mathrm{d}c}{\mathrm{d}t} = -qc, \quad c\,(t = 0) = c_0. \quad (10.2)$$

Step One: Solve the differential equation to find $c\,(t)$.

Step Two: When the pollutant concentration has reached 5% of its initial value we have $c\,(t) = 0.05c_0$. Denote the time at which this occurs by $t_{0.05}$.

Question 10.1 *Using the relationship $t_{0.05} = \frac{V}{q}\ln(20)$ calculate the time taken for a polutant to reach 5% of its initial value for the following lakes (Mesterton, 1989):*

***Lake Erie**: $V = 458 \times 10^9 m^3$ and $q = 480 \times 10^6 m^3\ day^{-1}$. (≈ 7.8 years)*

***Lake Ontario**: $V = 1636 \times 10^9 m^3$ and $q = 572 \times 10^6 m^3\ day^{-1}$. (≈ 23.5 years)*

10.2.2 Polutant dumped into a clean lake: Seasonal flow-rate

In section 10.2.1 assumed that the flow rate q is constant. However, in real-life the flow-rate is likely to be a function of time, i.e. $q(t)$.

Question 10.2 *Why is the flowrate likely to depend upon time?*

Question 10.3
*([**Barnes & Fulford**], **chapter 2.5**)*
At time $t = 0$ polutant is dumped into a clean lake giving an initial concentration of polutant c_0. Suppose that only fresh water flows into the lake and that the flowrate varies seasonaly with

$$q(t) = q_0 \left(1 + \epsilon \cos\left(\frac{2\pi t}{365}\right)\right).$$

Here q_0 is the mean flowrate and $-1 < \epsilon < 1$.

$$q(t) = q_0 \left(1 + \epsilon \cos\left(\frac{2\pi t}{365}\right)\right).$$

(a) Suppose that only fresh water flows into the lake. Find the sol^n of the model and derive an eq^n for the time it takes for the lake's pollution level to reach 5% of its initial valu

(b) What is the periodicity of the function $\epsilon \cos\left(\frac{2\pi t}{365}\right)$?

(c) Sketch the function

$$q(t) = q_0 \left[1 + \epsilon \cos\left(\frac{2\pi t}{365}\right)\right]$$

where $q_0 > 0$, $t \geq 0$ and $-1 < \epsilon < 1$ $(\epsilon \neq 0)$.

(d) On physical grounds explain why we have the restriction $-1 < \epsilon < 1$

From the question we have

$$c_{\text{in}} = 0,$$

$$q(t) = q_0 \left[1 + \epsilon \cos\left(\frac{2\pi t}{365}\right)\right].$$

Thus the model is

$$V \frac{\mathrm{d}c}{\mathrm{d}t} = -q_0 \left[1 + \epsilon \cos\left(\frac{2\pi t}{365}\right)\right] c, \quad (10.3)$$

$$c(t = 0) = c_0.$$

The solution to this differential equation is given by

When the pollutant concentration has reached 5% of its initial value we have $c(t) = 0.05 c_0$. Denote by $t_{0.05}$ the time at which this occurs. Then

Thus to find the value $t_{0.05}$ we must solve the equation

$$\ln(0.05) + \frac{q_0}{V}\left[t_{0.05} + \frac{365}{2\pi}\sin\left(\frac{2\pi t_{0.05}}{365}\right)\right] = 0.$$

We could solve this equation *numericaly* to find $t_{0.05}$ as a function of ϵ. Other approaches include

1. Re-arrange the equation to find q_0 as a function of $t_{0.05}$, i.e. $q_0 = f(t_{0.05})$, and plot the graph.
2. For small values of ϵ, $-1 \ll \epsilon \ll 1$ we can use a technique called *asymptotic expansions* to approximate the solution of the equation.

10.2.3 Polutant flowing into a lake

In sections 10.2.1 & 10.2.2 we assumed that only fresh water flows into a lake. This is not usualy the case. In this section the concentration of pollutant flowing into the lake is c_{in} and the concentration of pollutant in the lake at time $t = 0$ is c_0.

The model is

$$V\frac{\mathrm{d}c}{\mathrm{d}t} = qc_{\text{in}} - qc, \quad c(t=0) = c_0. \quad (10.4)$$

The solution of this equation is (see example 9.6)

$$c = c_{\text{in}} - (c_{\text{in}} - c_0)\, e^{-qt/V}. \quad (10.5)$$

Question 10.4

1. *What is the concentration of polutant in the lake as* $t \to \infty$*?*
2. *How does the concentration of polutant in the lake vary with time?* ***Hint***. *Consider the cases* $c_0 > c_{in}$, $c_0 < c_{in}$ *and* $c_0 = c_{in}$.
3. *Consider a lake of volume* $V = 4 \times 10^9 m^3$. *Water flows into the lake at a rate* $q = 4 \times 10^6\, m^3 day^{-1}$. *Effluent from a chemical plant flows into the lake with an inflow concentration of* $0.1 \times 10^{-9}\, kgm^{-3}$. *Suppose that the threshold of polutant for detection by the environment protection agency is* $0.2 \times 10^{-9}\, kgm^{-3}$.

   (a) *What concentration of polutant can be dumped into the lake on day 1 if there is to be an inspection in thirty days time?* $(0.203 \times 10^{-9}\, kgm^{-3})$.

   (b) *Convert your answer to the previous calculation to a mass of polutant. (**Hint** concentration = mass/Volume).*

10.2.4 A degradable polutant flowing into a lake

In this section we consider the model

$$V\frac{\mathrm{d}c}{\mathrm{d}t} = q\left(c_{\text{in}} - c\right) - Vk_1 c, \quad c\left(t=0\right) = c_0. \tag{10.6}$$

This equation arises when the pollutant species c degrades through a chemical reaction to a non-pollutant species d

$$c \to d. \tag{10.7}$$

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$$\boxed{V\frac{\mathrm{d}c}{\mathrm{d}t} = q\left(c_{\text{in}} - c\right) - Vk_1 c,\ c\left(t=0\right) = c_0.}$$

Equation 10.6 also arises if instead of a pollutant, c, flowing at a constant rate, q, into a lake of constant volume, V, we consider a chemical reactant, c, flowing at a constant rate, q, into a chemical reactor of constant volume, V.

The solution of equation 10.6 is

$$c = \frac{qc_{\text{in}}}{q+Vk_1} - \left[\frac{qc_{\text{in}}}{q+Vk_1} - c_0\right]\exp\left[-\frac{\left(q+Vk_1\right)}{V}t\right]. \tag{10.8}$$

$$c = \frac{qc_{\text{in}}}{q+Vk_1} - \left[\frac{qc_{\text{in}}}{q+Vk_1} - c_0\right] \exp\left[-\frac{(q+Vk_1)}{V}t\right].$$

Question 10.5 ***(first-order reaction in a chemical reactor)*** *Use equation (10.8) to answer the following questions. We define the dimensionless reactant concentration in the reactor as the quantity $C^* = c/c_{in}$.*

1. *In what units is the chemical rate k_1 measured?*
2. *What is the dimensionless reactant concentration C^* in the reactor as $t \to \infty$?*
3. *Define α by $\alpha = C^*(t = \infty)$. Sketch α as a function of the flowrate q. In chemical engineering applications we require α to be 'small'. What does this mean? How is this achieved?*
4. *Assuming that $c_0 = 0$ show that the time taken for C^* to reach the value 0.99α ($t_{0.99}$) is given by*

$$t_{0.99} = \frac{V}{q+k_1V}\ln(100)$$

5. *Why woud we use the number $t_{0.99}$?*

0.3 Extensions to the basic model

It is possible to extend the lake pollution models in various ways. One extension is to consider a network of lakes.

10.3.1 Two lakes in series

3.4.1.1 Rivers flowing into lake one only One extension of the basic problem is to consider two lakes, $\mathcal{L}_1$ and $\mathcal{L}_2$, in series with the outflow _______ from lake $\mathcal{L}_1$ flowing into lake $\mathcal{L}_2$. This situation is illustrated in figure 10.1

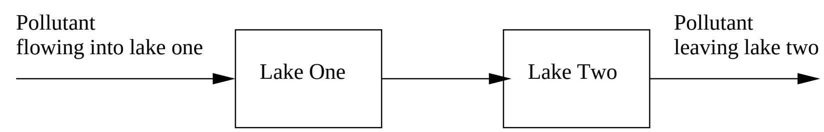

Figure 0.11: Schematic diagram of two lakes in series

Let c_i and V_i be the concentration of pollutant in lake i and the volume of lake i respectively. The model for the concentration of pollutant in the two lakes is

Concentration of pollutant in lake one

$$V_1 \frac{\mathrm{d}c_1}{\mathrm{d}t} = qc_{\mathrm{in}} - qc_1, \quad c_1\,(t=0) = (c_1)_0\,, \tag{10.9}$$

Concentration of pollutant in lake two

$$V_2 \frac{\mathrm{d}c_2}{\mathrm{d}t} = qc_1 - qc_2, \quad c_2\,(t=0) = (c_2)_0\,. \tag{10.10}$$

Note that the concentration of pollutant *entering* the second lake is equal to the concentration of pollutant *leaving* the first lake.

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A variation on this problem is important in chemical engineering, where we have a reactant flowing through two reactors, rather than a pollutant flowing through two lakes.

For instance, is it beter to have *one* reactor of volume V_0 or two reactors of volumes V_1 and V_2 with $V_1 + V_2 = V_0$?

The situation in which the reaction is catalysed by a biological agent is a topic of interest in bioreactor engineering. (For these problems the model has to include chemical reactions, just as we did in section 10.2.4).

Note You do *not* need to memorise the model equations. But you should be able to explain what any term in the model represents *physically*.

Question 10.6 *What do the terms* $+qc_{in}$ *and* $-qc_1$ *represent physically in equation* (10.9)*? What do the terms* $+qc_1$ *and* $-qc_2$ *represent in equation* (10.10)*?*

Concentration of polutant in lake one

$$V_1 \frac{\mathrm{d}c_1}{\mathrm{d}t} = qc_{\text{in}} - qc_1, \quad c_1\,(t=0) = (c_1)_0\,,$$

Concentration of polutant in lake two

$$V_2 \frac{\mathrm{d}c_2}{\mathrm{d}t} = qc_1 - qc_2, \quad c_2\,(t=0) = (c_2)_0\,.$$

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9.3.1.2 Rivers flowing into lakes one and two In section 9.3.1.1 we tacitly assumed that *all* the water flowing into lake two came from lake one. This makes sense for the analogous chemical engineering problem. However, for lakes this is not the case. It is probable that there are rivers flowing into lake two. This situation is illustrated in figure 10.2

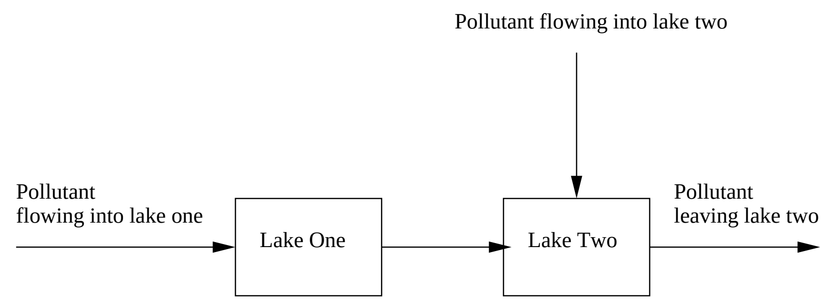
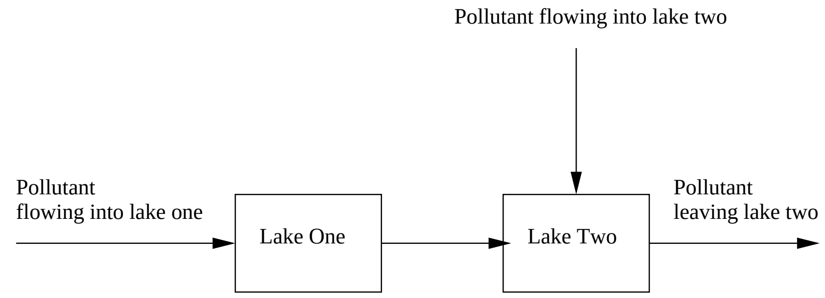


Figure0.12: Schematic diagram of two lakes in series

Let q_1 and q_2 be the flowrate of rivers flowing into lakes one and two respectively. Let $c_{\text{in},1}$ and $c_{\text{in},2}$ be the concentration of pollutant in the rivers flowing directly into lakes one and two respectively.

The model for the concentration of pollutant in the two lakes is



Concentration of polutant in lake one

$$V_1 \frac{\mathrm{d}c_1}{\mathrm{d}t} = q_1 c_{\text{in},1} - q_1 c_1, \quad c_1 (t=0) = (c_1)_0 \, , \tag{10.11}$$

Concentration of polutant in lake two

$$V_2 \frac{\mathrm{d}c_2}{\mathrm{d}t} = (q_1 c_1 - q_1 c_2) + (q_2 c_{\text{in},2} - q_2 c_2) \, , \tag{10.12}$$

$$c_2 (t=0) = (c_2)_0 \, ,$$

Note that the concentration of pollutant entering the second lake from the first lake is equal to the concentration of pollutant *leaving* the first lake.

Note You do *not* need to memorise the model equations. But you should be able to explain what any term in the model represents *physically*.

Question 10.7 *What do the terms* $+q_1c_1$, $-(q_1+q_2)\,c_2$ *and* $q_2c_{in,2}$ *represent in equation* (10.12)*?*

10.3.2 Three or more lakes

The situations considered in section 10.3.1 can be extended to cover the case of three, or more, lakes.

The equivalent chemical (bioreactor) engineering problem is of interest. (Again, we must add terms to the model to represent chemical reactions).

Comment 10.1 *From the perspective of chemical (bioreactor) engineering we are interested in questions such as, 'what arrangement of reactors give us the best performance?'* ***reflux?***

From the perspective of lake pollution we might ask the question 'An industrial company decides to dispose of a noxious substance of pumping it into a river. Given a choice of lake configurations which is the best option to minimise pollutant concentrations down stream?'

Rivers Eye-diseases.

0.4 Revision of key ideas

0.5 Concept map

Draw a concept map for this chapter relating the aims/key ideas of the chapter. If you are unfamiliar with the idea of a concept map see appendix A.