School of Mathematics & Applied Statistics

MATH111: Mathematics Applied Mathematical Modelling 1

Assignment Week 2 Solutions

Spring 2007

1. Give an example of

   (a) A third order nonlinear autonomous difference equation.

   (b) A fifth order linear nonautonomous difference equation.

   For each example, explain why your equations satisfies the stated criteria.

   Solution As each students answer should be different for this question no solution is provided.

2. The number of chickens in Mr & Mrs Tweedy's farm is modelled by the difference equation,

$$c_n = (1 + g - \alpha)\, c_{n-1} - P, \quad n = 1, 2, 3 \ldots$$

   where c_i is the number of chickens in the ith week, g is the fractional growth rate of chickens each week, α is the fraction of current chickens who are killed by foxes each week, and P is the constant number of chickens that are converted into pies each week. Assume that g and α are constant. For convenience let

$$\beta = 1 + g - \alpha,$$

   and suppose that in week 0 there are c_0 chickens present.

   (a) Find the general solution of the chicken model, simplifying as far as possible.
   Solution The equation is

$$c_n = \beta c_{n-1} - P.$$

   This is an equation of the form

$$x_n = a x_{n-1} + b\,(n)\,,$$

   the solution of which we know to be

$$x_n = a^n x_0 + \sum_{p=1}^{n} a^{n-p} b\,(p)\,.$$

   Hence

$$\begin{aligned} c_n &= \beta^n c_0 + \sum_{p=1}^{n} \beta^{n-p}\,(-P)\,, \\ &= \beta^n c_0 - P \sum_{p=1}^{n} \beta^{n-p}, \\ &= \beta^n c_0 - P \frac{(\beta^n - 1)}{\beta - 1}, \\ &= \beta^n c_0 - \frac{P\beta^n}{\beta - 1} + \frac{P}{\beta - 1}, \\ &= \left(c_0 - \frac{P}{\beta - 1}\right)\beta^n + \frac{P}{\beta - 1}. \end{aligned}$$

(b) Suppose that $c_0 = 200$, $g = 0.25$ and $\alpha = 0.05$.
Solution Note that $\beta = 1 + g - \alpha = 1 + 0.25 - 0.05 = 1.2$.

(i) Suppose that 20 chickens a week are converted into pies. What is the number of chickens on the farm in the limit $n \to \infty$?
Solution We have

$$\begin{aligned} c_n &= \left(200 - \frac{20}{0.2}\right)(1.2)^n + \frac{20}{0.2}, \\ &= 100\,(1.2)^n + 100. \end{aligned}$$

Thus

$$\lim_{n\to\infty} c_n = +\infty.$$

(ii) Suppose that 60 chickens a week are converted into pies. What is the number of chickens on the farm in the limit $n \to \infty$?
Solution We have

$$\begin{aligned} c_n &= \left(200 - \frac{60}{0.2}\right)(1.2)^n + \frac{60}{0.2}, \\ &= 300 - 100\,(1.2)^n. \end{aligned}$$

Thus

$$\lim_{n\to\infty} c_n = -\infty.$$

However, the question asks for the number of chickens on the farm and we can not have a negative number of chickens. Thus the answer is

$$\lim_{n\to\infty} c_n = 0.$$

(iii) What number of chickens (P) should be converted into pies each week if the number of chickens on the farm is to remain constant?
Solution We need the coefficient of β^n to be zero. Hence

$$\begin{aligned} c_0 - \frac{P}{\beta - 1} &= 0, \\ \Rightarrow P &= (\beta - 1)\,c_0, \\ &= 40. \end{aligned}$$

(iv) At the beginning of the first week the pie machine breaks down before any chickens are converted into pies. It will take Mr. Tweedy eight weeks to fix the pie machine. When it is fixed Mrs Tweedy will convert all the chickens into pies. If one chicken produces four pies how many pies will Mrs Tweedy have?
Solution Put $P = 0$ and $n = 8$ into the general solution to obtain

$$\begin{aligned} c_n &= c_0 \beta^n, \\ c_8 &= 200\,(1.2)^8, \\ &= 859 \quad \text{chickens (rounding down)} \\ \text{Pies} &= \text{chickens} \times 4, \\ &= 859 \times 4 = 3436. \end{aligned}$$

(v) How do the chickens feel about being converted into pies? What should they do?
Solution Left to your imagination.

3. Use induction to prove that

$$\sum_{k=1}^{n} a^{n-k}k = \frac{a^{n+1} - (n+1)\,a + n}{(a-1)^2}.$$

Simplify your expression as far as possible for the case when $a = -1$.

Solution

(i) Is the statement true for $n = 1$? The LHS of the statement gives

$$\sum_{k=1}^{1} a^{1-k}k = a^0 \times 1 = 1.$$

The RHS of the statement gives

$$\frac{a^2 - (2)\,a + 1}{(a-1)^2} = \frac{(a-1)^2}{(a-1)^2} = 1.$$

The statement is true for the case $n = 1$.

(ii) We now assume that the statement is true for the case n.

(iii) Is the statement true for the case $n+1$? The LHS of the statement gives

$$\begin{aligned}
\sum_{k=1}^{n+1} a^{n+1-k}k &= \sum_{k=1}^{n} a^{n+1-k}k + \sum_{k=n+1}^{n+1} a^{n+1-k}k, \\
&= a\sum_{k=1}^{n} a^{n-k}k + \sum_{k=n+1}^{n+1} a^{n+1-k}k, \\
&= a\left[\frac{a^{n+1} - (n+1)\,a + n}{(a-1)^2}\right] + (n+1)\,, \\
&= \frac{a^{n+2} - (n+2)\,a + (n+1)}{(a-1)^2}.
\end{aligned}$$

The RHS of the statement gives

$$\frac{a^{n+2} - (n+2)\,a + (n+1)}{(a-1)^2}$$

As LHS=RHS we conclude that the statement is true for all n.

When $a = -1$ we have

$$\begin{aligned}
\sum_{k=1}^{n} a^{n-k}k &= \frac{a^{n+1} - (n+1)\,a + n}{(a-1)^2}, \\
&= \frac{(-1)^{n+1} + (n+1) + n}{4}, \\
&= \frac{2n+1}{4} + \frac{(-1)^{n+1}}{4}, \\
&= \frac{2n+1}{4} + \frac{(-1)\,(-1)^n}{4}, \\
&= \frac{2n+1}{4} - \frac{(-1)^n}{4},
\end{aligned}$$

4. Consider the early-stages of a disease spreading through a population. Those individuals who have caught the disease are known as the ‘infectives’. Each week the following activities occur:

- Each infective infects a certain number of uninfected people.
- A fraction of the infectives recover naturally from the disease.
- A fraction of the infectives die as a result of the disease.
- A certain number of the infectives are treated and recover from the disease.

(a) Write down a **word** equation that defines this problem.
Solution Solution

$$\left\{\begin{array}{c}\text{Change in number}\\ \text{of infectives}\end{array}\right\} = \left\{\begin{array}{c}\text{infectives recovered}\\ \text{from the disease naturally}\end{array}\right\} - \left\{\begin{array}{c}\text{infectives who died}\\ \text{from the disease}\end{array}\right\} - \left\{\begin{array}{c}\text{infectives treated and}\\ \text{recovered from the disease}\end{array}\right\}.$$

(b) Write down, formally, the difference equation that describes the above scenario. Define **all** variables and explain your terms.
Solution Let I_w and I_{w-1} be the number of infectives in the population in weeks w and $w-1$ respectively.
Let u be the number of people infected by one infective person.
Let r be the fraction of infectives that recover naturally from the disease.
Let d be the fraction of infectives that die as a result of the disease. Let T be the number of infectives that are treated and recover from the disease.
Then

$$I_w - I_{w-1} = uI_{w-1} - rI_{w-1}dI_{w-1} - T,$$
$$\Rightarrow I_w = (1 + u - r - d)\, I_{w-1} - T.$$

5. In the mid-session test and/or the final exam you may be asked a question in which you need to write simple Maple code to solve a problem.

The following expression was derived by Golay (1958) for capillary chromatography with a retentive layer.

$$\Lambda = \frac{1 + 6\delta + 11\delta^2}{48\,(1+\delta)^3}.$$

In this equation Λ is the normalised dispersion, δ is a function of the capacitance ratio and the void fraction of the chromatography bed.

What is the maximum value of the normalised dispersion, Λ_{max}, and what is the corresponding value of δ?

M.J.E. Golay (1958). Theory of chromatography in open and coated tubular columns with round and rectangular cross-sections, in Gas Chromatography, D.H. Desty, ed., Butterworth, London, pp 36-49.

Solution There is more than one way to answer this question. The value for δ that maximise Λ and the corresponding value for Λ are given by

$$\delta = \frac{5+\sqrt{58}}{11}, \qquad = 1.147,$$
$$\Lambda = \frac{121\left(31 + 4\sqrt{58}\right)}{12\left(16+\sqrt{58}\right)^3} \qquad = 0.047.$$

Here's my maple code for this question.

```
#week2-2007.maple
#09.08.07
#
Lambda := (1+6*delta+11*delta^2)/(48*(1+delta)^3); # define the function
Ld     := diff(Lambda,delta);  # Find the first derivative
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```
Ldd    := diff(Ld,delta);      # Find the second derivative

solve(Ld,delta);               # solve the equation. There are two solutions!

# Find the value for Lambda when
# delta = (5-sqrt(58))/11 and check to see if it is a maximum or a minimem

delta := (5-sqrt(58))/11;
simplify(Lambda);  # note use of simplify command to make the expression
                   # ‘nicer’ to look at.
evalf(Ldd);        # note use of evalf to obtain a numeric answer.

# Ldd = 0.94 so this is a MINIMUM value.

# Find the value for Lambda when
# delta = (5+sqrt(58))/11 and check to see if it is a maximum or a minimem

delta := (5+sqrt(58))/11;
simplify(Lambda);  # note use of simplify command to make the expression
                   # ‘nicer’ to look at.
# note use of evalf command in the following to obtain a numeric answer.
evalf(Lambda);
evalf(Ldd);
evalf(delta);

# Ldd = -0.01 so this is a MINIMUM value.
```