School of Mathematics & Applied Statistics

MATH111: Mathematics Applied Mathematical Modelling 1

Assignment Week 10

Spring 2007

Student Name: ______________________ *Student Number:* ____________

FULL WORKING is to be shown for all solutions.
Untidy or badly set out work will not be marked and will be recorded as unsatisfactory.
This assignment is to be handed in during your tutorial in Week 11

The assignment that you had in *must* include a cover page. On the cover-page you should briefly answer the following questions

(a) What topic did you believe was the most important in the assignment?

(b) Why do you believe that is the most important topic?

(c) What problems did you have with the assignment, if any?

You should answer each question with a complete sentence.

If you fail to provide a cover-page your assignment will automatically be marked 'unsatisfactory'.

You may choose to answer one of the questions on this assignment sheet by working in a group of upto four individuals. If you choose this option then at the end of your group answer you must list the members of your group.

School of Mathematics & Applied Statistics **MATH111: Applied Mathematical Modelling 1**
Assignment Week 10
Submission Receipt

Student Name: ______________________ *Student Number*: ____________

Tutorial Class: ________ *Date Submitted:* ____________ *Tutor Initials:* ____________

1. The population of fish is modelled by the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x),$$

where the function $f(x)$ is given by

$$f(x) = rx\left(1 - \frac{x}{K}\right)\left(\frac{x}{K_0} - 1\right),$$

where $r > 0$ and $0 < K_0 < K$.

   (a) Sketch the growth curve $f(x)$ as a function of x.

   (b) Using your sketch determine the stability of the steady-state solutions $x = 0$, $x = K_0$ and $x = K$, carefully explaining your reasoning.

   (c) Calculate the stability of the steady-state solutions $x = 0$, $x = K_0$ and $x = K$ by finding their eigenvalues.

   (d) How does the long-term evolution of the differential equation depend upon the choice of the initial condition x_0?

   (e) A disease spreads through the population reducing the population density to a value a. What happens to the population? Justify your answer.

2. The population density of the spruce budworm in a forest is given by the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = rx\left(1 - \frac{x}{q}\right) - \frac{x^2}{1 + x^2}.$$

Determine the stability of the trivial steady-state solution $x = 0$.

3. The logistic equation with constant effort harvesting is given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = rx\left(1 - \frac{x}{K}\right) - Ex, \quad x(0) = K,$$

where x is the population density of an animal in an environment, t time, $r > 0$ is the intrinsic growth rate, $K > 0$ is the carrying capacity of the environment and $E \geq 0$ is the effort expanded in harvesting.

   (a) Determine the steady-state solutions of the logistic equation with constant effort harvesting.

   (b) Suppose that the value for the intrinsic growth rate is $r = 3$.

      (i) Determine the stability of the steady-state solutions as a function of the effort E.

      (ii) Draw a steady-state diagram for the logistic equation with constant effort harvesting for the case $r = 2$ showing how the steady-state solutions of the model vary as a function of the effort expanded in harvesting. Indicate stable and unstable steady-state solutions using solid and dashed lines respectively.

      (iii) For what value of the parameter E does a bifurcation occur? What kind of bifurcation is it?

4. I think that it is important that you develop an appreciation for some of the applications of mathematics in the 'real world'. The following question is designed to help you do this. It is based on Cipra (2007). You only need to write a couple of sentences to answer each of the following questions.

B.A. Cipra. 2007. Geosciences conference tackles global issues. *SIAM News*, **40**(5), pages 1, 8& 9. `www.siam.org/pdf/news/1132.pdf`

If you would like to do additional reading a good place to start is the conference web page: `www.siam.org/meetings/gs07/`.

   (a) Explain how mathematical modelling can be used to enhance oil recovery. ('Simple, Flexible Models of Subsurface Flow').

   (b) What is 'earthquake inversion'? Why is it important? What type of mathematical problem is it? ('Earthquake Inversion — Algorithmic Challenges')

(c) Explain one application of mathematical modelling in investigating the feasibility of carbon dioxide sequestration. ('The Modeling Problem of a Lifetime')

In the the final exam you may be asked a question about Maple.

Your answer should include all maple code that you used to obtain the answer.

5. Thermal and solutal dispersion in a circular tube with diffusion into the wall is characterised by a dispersion coefficient (Λ) which is a function of the void fraction of the bed (ϵ_f) and the ratio of fluid thermal diffusivity to diffusivity in the wall (μ).

   Balakotaiah and Chang (2003) obtained the following formula for the dispersion coefficient

$$\Lambda = \frac{1}{48} g_1(\epsilon_f) + \frac{\mu}{8} g_2(\epsilon_f),$$
$$g_1(\epsilon_f) = \epsilon_f\left(6\epsilon_f^2 - 16\epsilon_f + 11\right),$$
$$g_2(\epsilon_f) = \epsilon_f\left[4\epsilon_f - \epsilon_f^2 - 3 - 2\ln(\epsilon_f)\right].$$

   (a) What is the maximum value of the function g_1 and for what value of the void fraction (ϵ_f) does it occur?

   (b) What is the maximum value of the function g_2 and for what value of the void fraction (ϵ_f) does it occur?

   (c) What is the maximum value of the dispersion coefficient (Λ) and the corresponding value for the void fraction (ϵ_f) when the ratio of fluid thermal diffusivity to diffusivity in the wall (μ) is equal to ten?

   (d) Plot the dispersion coefficient (Λ) as a function of the void fraction (ϵ_f) when the ratio of fluid thermal diffusivity to diffusivity in the wall (μ) is equal to 1.

V. Balakotaiah & Chang, H-C. (2003). Hyperbolic homogenized models for thermal and solutal dispersion. *SIAM Journal on Applied Mathematics*, **63**(4), 1231–1258.