

School of Mathematics & Applied Statistics
**MATH111: Mathematics Applied Mathematical
 Modelling 1**
Assignment Week 2 Solutions
Spring 2006

1. For the following difference equations: identify the state variable and the ‘time’ variable; give their order and state whether they are linear, nonlinear, autonomous or non-autonomous.

(a) $n_{y+2} = n_{y+1}y$

(b) $y_{n+1} = 2y_{n-1} + \sin(n)$

Solution.

State variable.

In (a) the state variable is n . In (b) the state variable is y .

Time variable.

In (a) the ‘time’ variable is y . In (b) the ‘time’ variable is n .

Order.

If we denote P and Q to be the largest and smallest subscripts on the variable that occur in the difference equation, then the **order** of the difference equation is given by $P - Q$.

In (a) the order is $(y + 2) - (y + 1) = 1$.

In (b) the order is $(n + 1) - (n - 1) = 2$.

Linear or non-linear?

A difference equation is said to be linear if each of the states $\dots, x_{n-2}, x_{n-1}, x_n, x_{n+1}, x_{n+2}, \dots$ appearing in the equation are present linearly. Otherwise it is termed **non-linear**.

In (a) n_{y+2} and yn_{y+1} are both linear terms. Thus the equations is linear.

In (b) y_{n+1} and $2y_{n-1}$ are both linear. Thus the equation is linear — note we do not examine the term $\sin(n)$ as this expression does not contain a state variable.

Autonomous or non-autonomous?

If a difference equation contains the ‘time’ variable explicitly in the equation, then we refer to it as a non-autonomous difference equation. Otherwise it is called **autonomous**.

In (a) the subscript is y . This appears in the equation as $n_{y+1}y$. Thus the equation is non-autonomous.

In (b) the subscript is n . This appears in the equation as $\sin(n)$. Thus the equation is non-autonomous.

2. (a) Show that $x_n = an + b$ is a solution of the difference equation

$$x_{n+1} - 2x_n + x_{n-1} = 0,$$

where a and b are constants.

- (b) Find the solution of the difference equation

$$n_{x+1} - 2n_x + n_{x-1} = 0, \quad n(x=1) = 7, \quad n(x=3) = 13.$$

Solution

- (a) From the proposed we have

$$x_{n+1} = a(n+1) + b,$$

$$x_n = an + b,$$

$$x_{n-1} = a(n-1) + b.$$

Substituting these expressions into the LHS of the difference equation we have

$$\begin{aligned} x_{n+1} - 2x_n + x_{n-1} &= a(n+1) + b - 2(an+b) + a(n-1) + b, \\ &= (an - 2an + an) + (a - a) + (b - 2b + b), \\ &= 0, \\ &= \text{RHS. Thus } x_n = an + b \text{ is a solution of the difference equation.} \end{aligned}$$

(b) From our answer to part (b) we know that the general solution is given by

$$n_x = ax + b .$$

From the information in the question we have

$$\begin{aligned} n_1 &= a + b = 7, \\ n + 3 &= 3a + b = 13. \end{aligned}$$

It follows that

$$a = 3 \quad b = 4.$$

The solution of the difference equation is therefore

$$n_x = 3x + 4.$$

3. Solve the following difference equations to obtain solutions in “closed form”.

(a) $y_{n+1} - \frac{1}{2}y_n = 2, y_0 = c.$

(b) (i) Evaluate the expression $\sum_{p=1}^n 2^{n-p} \cdot 3^p$. Hint Show that

$$\sum_{p=1}^n 2^{n-p} \cdot 3^p = 2^n \left[\left(\frac{3}{2}\right) + \left(\frac{3}{2}\right)^2 + \dots + \left(\frac{3}{2}\right)^n \right].$$

(ii) Hence solve the difference equation $x_n = 2x_{n-1} + \frac{3^n}{3}, x_0 = 0.5.$

Solution.

The solution to the equation

$$x_n = ax_{n-1} + b(n)$$

is

$$x_n = a^n x_0 + \sum_{p=1}^n b(p) a^{n-p}.$$

(a) $y_{n+1} - \frac{1}{2}y_n = 2, y_0 = c.$ Put this equation in the general form by substituting $n \rightarrow n - 1$. Then the equation becomes $y_n - \frac{1}{2}y_{n-1} = 2, y_{-1} = c.$ We have

$$\begin{aligned} a &= \frac{1}{2}, \\ b(n) &= 2. \end{aligned}$$

Thus the solution is

$$\begin{aligned} y_n &= \left(\frac{1}{2}\right)^n y_0 + \sigma_{p=1}^n 2 \left(\frac{1}{2}\right)^{n-p}, \quad \text{where } y_0 \text{ is the initial condition,} \\ &= \left(\frac{1}{2}\right)^n y_0 + 2\sigma_{p=1}^n \left(\frac{1}{2}\right)^{n-p}, \\ &= \left(\frac{1}{2}\right)^n y_0 + 2 \cdot 1 \left[\frac{\left(\frac{1}{2}\right)^n - 1}{\frac{1}{2} - 1} \right], \end{aligned}$$

(as the term inside the summation sign is a geometric progression with $k = 1$ and $r = \frac{1}{2}$).

$$= 4 + \left(\frac{1}{2}\right)^n (y_0 - 4).$$

Now we have

$$\begin{aligned} y_0 &= \frac{1}{2}y_1 + 2, \\ &= \frac{1}{2}c + 2. \end{aligned}$$

Thus the solution is

$$y_n = 4 + \left(\frac{1}{2}\right)^n \left(\frac{1}{2}c - 2\right).$$

(b) (i)

$$\begin{aligned} \mathcal{I} &= \sum_{p=1}^n 2^{n-p} \cdot 3^p, \\ &= 2^{n-1} \cdot 3 + 2^{n-2} \cdot 3^2 + \dots + 2^1 \cdot 3^{n-1} + 2^0 \cdot 3^n, \\ &= 2^n (2^{-1} \cdot 3 + 2^{-2} \cdot 3^2 + \dots + 2^{n-1} \cdot 3^{n-1} + 2^{-n} \cdot 3^n), \\ &= 2^n \left[\left(\frac{3}{2}\right) + \left(\frac{3}{2}\right)^2 + \dots + \left(\frac{3}{2}\right)^{n-1} + \left(\frac{3}{2}\right)^n \right]. \end{aligned}$$

Note that the term inside the brackets is a geometric progression with $k = \frac{3}{2}$ and $r = \frac{3}{2}$. It's sum is $3 \left[\left(\frac{3}{2}\right)^n - 1\right]$. Thus

$$\mathcal{I} = 3 \cdot 2^n \left[\left(\frac{3}{2}\right)^n - 1 \right].$$

(ii)

$$\begin{aligned} a &= 2, \\ b(n) &= \frac{3^n}{3}, \\ x_0 &= 0.5. \end{aligned}$$

Thus the solution is

$$\begin{aligned} x_n &= (2)^n \cdot \frac{1}{2} + \sum_{p=1}^n 1^n (2)^{n-p} \frac{3^p}{3}, \\ &= 2^{n-1} + \frac{1}{3} \sum_{p=1}^n 1^n (2)^{n-p} 3^p, \\ &= 2^{n-1} + 2^n \left[\frac{3^n}{2} - 1 \right], \\ &= 2^{n-1} + 3^n - 2^n, \\ &= 3^n - 2^{n-1}. \end{aligned}$$

4. Consider the problem of modelling the number of chickens in Mr & Mrs Tweedy's farm. Each week the following activities occur:

- The number of chickens increases through natural growth by 10%.
- A fraction, α , of the chickens are killed by foxes.

- A constant number of chickens are converted into chicken pies.

- (a) Write down a **word** equation that defines this problem.
- (b) Write down, formally, the difference equation that describes the above scenario. Define **all** variables and explain your terms.

Solution

(a)

$$\left\{ \begin{array}{c} \text{Change in} \\ \text{chicken numbers} \end{array} \right\} = \left\{ \begin{array}{c} \text{net} \\ \text{growth} \end{array} \right\} - \left\{ \begin{array}{c} \text{chickens killed} \\ \text{by foxes} \end{array} \right\} - \left\{ \begin{array}{c} \text{chickens converted} \\ \text{to pies} \end{array} \right\}$$

- (b) Let C_w and C_{w-1} be the number of chickens on the farm in weeks w and $w - 1$ respectively. Let g be the net growth rate. Let α be the fraction of chickens that are killed by foxes. Let N be the number of chickens that are converted into pies.

Then

$$\begin{aligned} C_w - C_{w-1} &= gC_{w-1} - \alpha C_{w-1} - N, \\ \Rightarrow C_w &= (1 + g - \alpha) C_{w-1} - N. \end{aligned}$$

The question states that $g = 0.1$. Thus

$$C_w = (1.1 - \alpha) C_{w-1} - N.$$

5. Municipal solid waste (MSW) may contain upto 30-40% of organic materials by mass. These organic wastes should be removed from the MSW before the MSW is delivered to a landfill site. One way to do this is to biologically oxidise the organic fraction.

The maximum specific oxidation rate (hr^{-1}) in a bioreactor is given by the formula

$$\mu_{\max} = \frac{A' \exp \left[-\frac{E_g}{RT} \right]}{1 + B \exp \left[-\frac{\Delta G_d}{RT} \right]},$$

where A' (hr^{-1}) and B ($-$) are constants, E_g is the activation energy of the growth process (kJ/mol), and ΔG_d is the Gibbs free energy change upon protein denaturation (kJ/mol).

In (1) the following parameter values were estimated: $A' = 4.032 \times 10^8 \text{ hr}^{-1}$, $B = 4.776 \times 10^{89}$, $E_g = 56.861 \text{ kJ/mol}$, $\Delta G_d = 537.56 \text{ kJ/mol}$, $R = 8.31431 \text{ kJ/mol/K}$.

- (1) E. Liwarska-Bizukojc, M. Bizukojc and S. Ledakowicz. 2001. Kinetic model for the process of aerobic biodegradation of organic fraction of municipal solid waste. *Bioprocess and Biosystems Engineering*, **24**: 195–202.

- (a) Plot the function μ_{\max} as a function of temperature over the range $275 \leq T \text{ (K)} \leq 320$.
- (b) Find the value of T (to one decimal place) that maximises the value of μ_{\max} .

Solution.

- (a) Here's my maple code for both parts of this question and the figure it generated.

```
# week2-2006.maple (i) Plot the maximum specific oxidation rate
# 29.07.06          as a function of temperature.
#                  (ii) Find the value of T that maximum the oxidation rate.
#
# Define the function
#
mumax := A*exp(-Eg/(R*T))/ (1+B*exp(-Gd/(R*T)));

# Define the constants
```

```

A := 4.032e8;
B := 4.776e89;
Eg := 56.861e3;
Gd := 537.56e3;
R := 8.31431;

# PART (i) plot the function

plot(mumax,T=275..320,labels=["Temperature","Maximum specific oxidation rate"],\
labeldirections=[horizontal,vertical]);

# PART (ii) differentiate the function. Then solve the equation to find T.

eqn := diff(mumax,T);
solve(eqn,T);

```

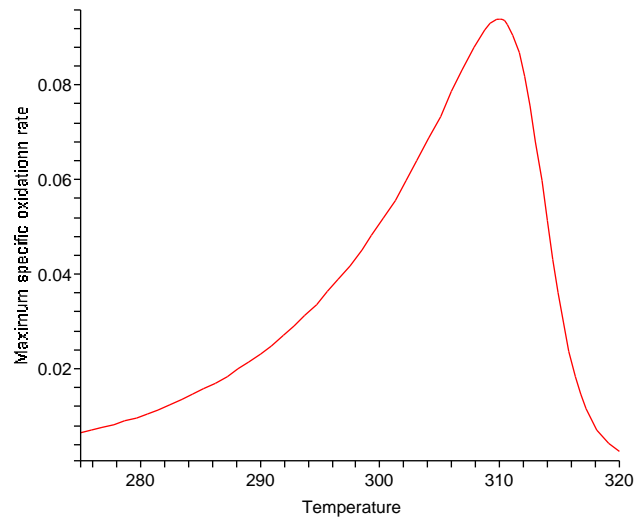


Figure 1: Figure for question (5i) showing the maximum specific oxidation rate as a function of temperature.

(b) $T = 309.9$ (K).