School of Mathematics & Applied Statistics

MATH11: Mathematics Applied Mathematical Modelling 1

Assignment Week 10

— Solutions

Spring 2004

1. The population density of fish is modelled by the differential equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = f(u), \qquad u(t=0) = u_0,$$

where the function $f(u)$ has the following properties:

- $f(0) = f(K_0) = f(K) = 0$ where $0 < K_0 < K$.
- If $u \in (0, K_0)$ then $f(u) < 0$.
- If $u \in (K_0, K)$ then $f(u) > 0$.
- If $u > K$ then $f(u) < 0$.

(a) Sketch the growth curve $f(u)$ as a function of u.
Answer See figure 1

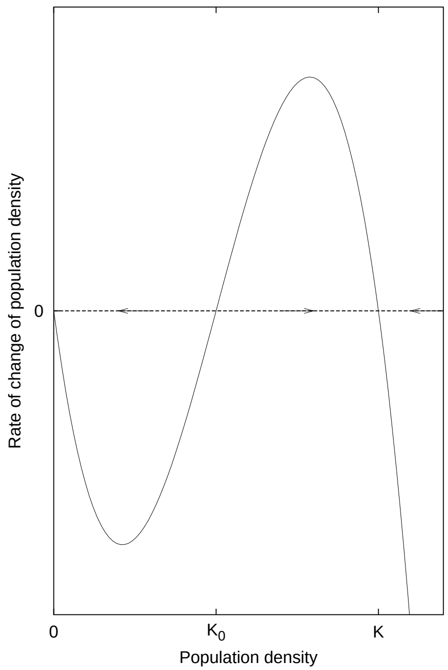

Figure 1: Sketch of the function described in question 1 (a).

(b) Using your sketch determine the stability of the steady state solutions $u = 0$, $u = K_0$ and $u =$ carefully explaining your reasoning.
Answer
The steady-state $\underline{u = 0 \text{ is } stable}$. If we change the initial condition by a small amount from $u_0 =$ $u_0 = 0 + \epsilon$, where $\epsilon > 0$, the rate of change of population density is negative. Thus the solution u will decrease towards the steady-state solution $u = 0$.
The steady-state $\underline{u = K_0 \text{ is } unstable}$. If we change the initial condition by a small amount fr $u_0 = K_0$ to $u_0 = K_0 \pm \epsilon$, where $\epsilon > 0$, the rate of change of population density is either nega $(u_0 = K_0 + \epsilon)$, or positive $(u_0 = K_0 - \epsilon)$. Thus the solution $u(t)$ will either decrease towards steady-state solution $u = 0$ $((u =_0 K_0 + \epsilon))$ or increase towards the solution $u = K$ $(u_0 = K_0 - \epsilon)$
The steady-state $\underline{u = K \text{ is } stable}$. If we change the initial condition by a small amount from $u_0 = K$ $u_0 = K \pm \epsilon$, where $\epsilon > 0$, the rate of change of population density is either negative $(u_0 = K_0 + \epsilon)$ positive $(u_0 = K_0 - \epsilon)$. Thus the solution $u(t)$ will either decrease towards the steady-state solut $u = K$ $(u_0 = K_0 + \epsilon)$ or increase towards the steady-state solution $u = K$ $(u_0 = K_0 - \epsilon)$.

(c) How does the long-term evolution of the differential equation depend upon the choice of the ini condition u_0?
Answer

- If $\underline{u_0 = 0}$ then
$$u(t) = 0\,\forall t.$$
- If $\underline{u_0 \in (0, K_0)}$ then
$$\lim_{t\to\infty} u(t) = 0.$$
Moreover the population density decreases monotonically towards the steady-state solution $u =$
- If $\underline{u_0 = K_0}$ then
$$u(t) = K_0\,\forall t.$$
- If $\underline{u_0 \in (K_0, K)}$ then
$$\lim_{t\to\infty} u(t) = K.$$
Moreover the population density increases monotonically towards the steady-state solution $u =$
- If $\underline{u_0 = K}$ then
$$u(t) = K\,\forall t.$$
- If $\underline{u_0 > K}$ then
$$\lim_{t\to\infty} u(t) = K.$$
Moreover the population density decreases monotonically towards the steady-state solut $u = K$.

(d) A disease spreads through the population reducing the population to a density $K_0/2$. What happ to the population? Justify your answer.
Answer The population will become *extinct* because the initial condition is

$$u_0 = K_0/2 \in (0, K).$$

The result follows from our answer to the previous question.

2. For some organisms finding a suitable mate may cause difficulties at low population densities, and a m realistic equation for population growth under these conditions may be

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN^2, \quad r > 0, N(0) = N_0.$$

(a) Solve this problem and show that the solution becomes infinite in finite time.

Answer

$$
\begin{aligned}
\frac{\mathrm{d}N}{N^2} &= r\mathrm{d}t \\
\int_{N_0}^{N} \frac{\mathrm{d}N}{N^2} &= \int_0^t r\mathrm{d}t \\
\left[\frac{-1}{N}\right]_{N_0}^{N} &= rt \\
\frac{-1}{N} + \frac{1}{N_0} &= rt \\
\frac{1}{N} &= \frac{1}{N_0} - rt \\
\frac{1}{N} &= \frac{1}{N_0} - \frac{rtN_0}{N_0} \\
N &= \frac{N_0}{1 - rtN_0}.
\end{aligned}
$$

The solution becomes *infinite* when $t = \dfrac{1}{rN_0}$. □

(b) The model above may be improved to

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN^2\left(1 - \frac{N}{K}\right).$$

Without integrating this equation find the steady-state solutions and say whether they are stable or unstable. (Do not calculate eigenvalues).

Answer The steady-state solutions of the differential equation

$$\frac{\mathrm{d}N}{\mathrm{d}t} = f(N)$$

are found by solving the equation

$$f(N) = 0.$$

The solutions of the equation

$$rN^2\left(1 - \frac{N}{K}\right) = 0$$

are clearly $\underline{N = 0}$ and $\underline{N = K}$.

The rate of change of population density as a function of the population density is sketched in figure 2. By considering the sign of the derivative function near the two steady-state solutions we see that the steady-state solution $\underline{N = 0}$ *is unstable* whilst the steady-state solution $\underline{N = K}$ *is stable*. □

(c) Derive the model of part (b) from that of part (a). (Hint...reread Chapter 11).

Answer Consider a simple population model in which the birth and death rates are given by βN^2 and αN^2 respectively. Then the associated population model is

$$
\begin{aligned}
\frac{\mathrm{d}N}{\mathrm{d}t} &= \beta N^2 - \alpha N^2 \\
&= (\beta - \alpha) N^2 \\
&= rN^2, \quad \text{where} \quad r = \beta - \alpha.
\end{aligned}
$$

This is the model analysed in part (a) of this question.

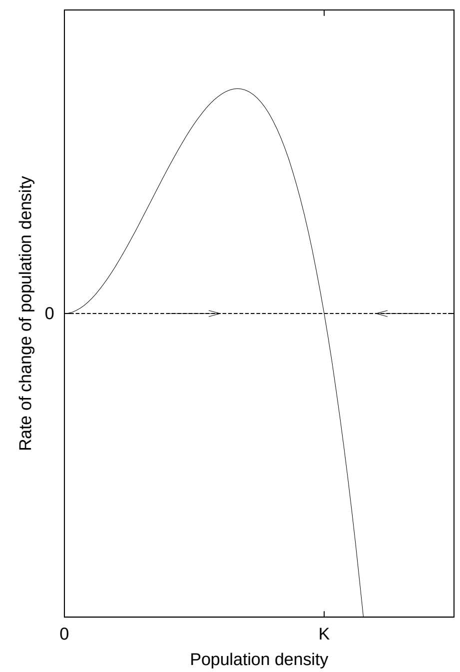

Figure 2: Sketch for question 2 (b).

In deriving this equation we have assumed that the birth rate increases with the square of the p ulation density. However, at *high population densities* the birth rate should decrease because competition for resources. A better assumption may be to model the birth rate by the funct $\beta(1 - aN)N^2$. This leads to the differential equation

$$
\begin{aligned}
\frac{\mathrm{d}N}{\mathrm{d}t} &= \beta(1 - aN)N^2 - \alpha N^2 \\
&= [\beta(1 - aN) - \alpha] N^2 \\
&= [(\beta - \alpha) - a\beta N] N^2 \\
&= (\beta - \alpha)\left[1 - \frac{a\beta}{\beta - \alpha} N\right] N^2 \\
&= r\left(1 - \frac{N}{K}\right) N^2,
\end{aligned}
$$

where $r = \beta - \alpha$ and $K = \dfrac{\beta - \alpha}{a\beta}$.

3. Suppose a population satisfies a logistic model with carrying capacity 100 and that the population siz 10 when $t = 0$ and 20 when $t = 1$. Find the intrinsic growth rate.

Use the solution to the logistic equation

$$x(t) = \frac{Kx_0}{x_0 + (K - x_0)e^{-rt}}.$$

Answer We have

$$K = 100, x_0 = 10, x(1) = 20.$$

Thus the value of r is obtained from

$$\begin{aligned}
20 &= \frac{100 \cdot 10}{10 + (100 - 10)\, e^{-r \cdot 1}} \\
20 &= \frac{100 \cdot 10}{10 + 90 e^{-r \cdot 1}} \\
10 + 90 e^{-r} &= 50 \\
90 e^{-r} &= 40 \\
e^{-r} &= \frac{4}{9} \\
\frac{9}{4} &= e^{r} \\
r &= \ln \frac{9}{4}. \\
&\approx 0.8109 \quad \text{(4 dp)}
\end{aligned}$$

□

4. Nisbet & Gurney (983) suggested the following form for the per-capita growth rate

$$r(x) = r \exp\left[1 - \frac{x}{K}\right] - d$$

Consider the associated population model

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \left(r \exp\left[1 - \frac{x}{K}\right] - d\right) x, \quad x(0) = x_0,\, r > de^{-1}.$$

(a) Find the steady-state(s) of the model. How do the number of steady-state solutions ($x^* \geq 0$) and their biological feasibility depend upon the values of r and d?

Answer The steady-state solutions for the equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x)$$

are the values of x, x^*, for which

$$\begin{aligned}
f(x) &= 0. \\
\text{i.e.} \quad \left(r \exp\left[1 - \frac{x}{K}\right] - d\right) x &= 0.
\end{aligned}$$

The solutions of this equation are given by

$$x^* = 0$$

and

$$\begin{aligned}
r \exp\left[1 - \frac{x^*}{K}\right] - d &= 0 \\
&\Rightarrow \exp\left[1 - \frac{x^*}{K}\right] = \frac{d}{r} \\
&\Rightarrow 1 - \frac{x^*}{K} = \ln\left(\frac{d}{r}\right) \\
&\Rightarrow \frac{x^*}{K} = 1 - \ln\left(\frac{d}{r}\right) \\
&\Rightarrow x^* = K\left[1 - \ln\left(\frac{d}{r}\right)\right]
\end{aligned}$$

There are *two* steady-state solutions: $x^* = 0$, and $x^* = K\left[1 - \ln\left(\frac{d}{r}\right)\right]$. The second of these is c biologically feasible if

$$\begin{aligned}
K\left[1 - \ln\left(\frac{d}{r}\right)\right] &> 0 \\
\Rightarrow \quad 1 &> \ln\left(\frac{d}{r}\right) \\
\Rightarrow \quad e &> \frac{d}{r}
\end{aligned}$$

(b) Explain why it is reasonable to assume that $r > de^{-1}$.

If $de^{-1} > r$ then there is only one biologically feasible steady-state solution, $x^* = 0$. A graph of function $f(x)$ shows that in this case the population becomes extinct. This is not very interesti So, let us suppose that $de^{-1} < r$.

(c) Sketch $\frac{\mathrm{d}x}{\mathrm{d}t}$ as a function of x. Hence determine how the long-term dynamics of the model depe upon the initial value x_0.

Answer The function is sketched in figure 3.

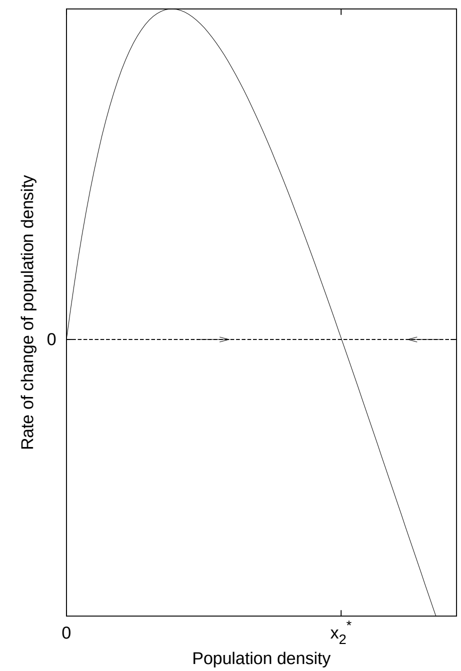

Figure 3: Sketch of the rate of change of population density for the Nisbet & Gurney model.

Let $x_1^* = 0$ and $x_2^* = K\left[1 - \ln\left(\frac{d}{r}\right)\right]$.

(i) If $x(0) = 0$ then $x(t) = 0 \quad \forall t$.
If $x(0) = x_2^*$ then $x(t) = x_2^* \quad \forall t$.

(ii) If $x(0) \in (0, x_2^*)$ then $\lim_{t \to \infty} x(t) = x_2^*$.

(iii) If $x(0) > x_2^*$ then $\lim_{t \to \infty} x(t) = x_2^*$.

□