D Taylor series

D.1 Taylor series expansions

Values of polynomials, such as $p(x) = x^3 + 2x^2 - 3x + 1$, may be readily calculated for any value of x whereas many other functions, such as $f(x) = \sin x$, cannot be evaluated, for most values of x, without the aid of a calculator.

In section D.2 we show how to approximate a function f near a given point a by a polynomial.

D.2 Taylor series expansion

The Taylor series of degree n that approximates a function f about the point $x = a$ is given by

$$\begin{aligned} f(x) &= f(a) + (x-a) f'(a) \\ &\quad + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) \\ &\quad + \cdots + \frac{(x-a)^n}{n!} f^{(n)}(a) \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x-a)^k, \end{aligned}$$

where $f^{(0)}(x) = f(x)$.

This equation

$$f\left(x+a\right)=\sum_{k=0}^{n}\frac{f^{(k)}\left(a\right)}{k!}\left(x-a\right)^{k}.$$

provides an extremely accurate polynomial approximation for a large class of functions. With reference to figure D.1, close to $x = x_1$ we can approximate the curve $f(x)$ with the line tangent to the curve at $x = x_1$. The approximation is not so good further away from $x = x_1$.

It should be noted that when finding the nth Taylor polynomial, the 'n' refers to the *degree of the highest term*, and not to the *number of terms.*

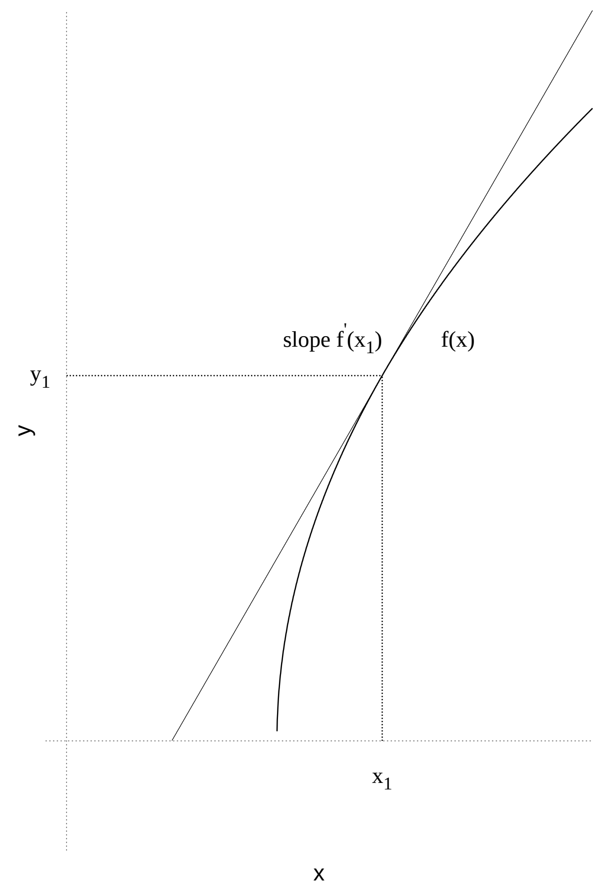

Figure D.1: Diagram showing the first-order Taylor expansion in the vicinity of the point x_1.

Example

1. Suppose $f(x) = \ln(x)$ with $a = 1$.

$$f(1) = \ln(1) = 0$$

$$f'(x) = \_ \qquad f'(1) = \_$$

$$f''(x) = \_\_\_ \qquad f''(1) = \_\_\_$$

$$f'''(x) = \_\_ \qquad f'''(1) = \_$$

Thus we can approximate the function $f(x) = \ln(x)$ near the point $x = 1$ by the following sequence of functions.

$$p_1(x) = \_\_\_\_\_\_\_\_\_\_$$

$$p_2(x) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

$$p_3(x) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

$$\_\_\_\_\_\_\_\_\_\_$$

$$0 + (x - 1)$$

$$0 + (x - 1) - \frac{1}{2!}(x - 1)^2$$

$$0 + (x - 1) - \frac{1}{2!}(x - 1)^2 + \frac{2}{3!}(x - 1)^3$$

$$\frac{1}{x}$$
$$\frac{-1}{x^2}$$
$$\frac{2}{x^3}$$

Taylor series

Each successive approximation is better than the previous one.

Suppose that we want to approximate the value of $\ln(1.1)$. The values of the three polynomials given above evaluated at $x = 1.1$ are

$$p_1(1.1) = 0.1$$
$$p_2(1.1) = 0.095$$
$$p_3(1.1) = 0.095333$$

respectively whereas the exact value, correct to six decimal places, is 0.095308.

2. Find the first two terms in the Taylor series approximation to $f(x) = \sin(x)$ near the point $a = 0$.

$$\sin x = \sin 0 + x \sin' 0 + \frac{x^2}{2} \sin'' 0 + \frac{x^3}{6} \sin''' 0 + \dots$$

$$= \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \quad \_\_\_\_\_\_\_\_\_\_\_\_$$

$$= \_\_\_\_\_\_\_\_\_\_$$

$$\sin 0 + x \cos 0 - \frac{x^2}{2} \sin 0 - \frac{x^3}{6} \cos 0 + \dots$$

$$x - \frac{x^3}{6} + \dots$$