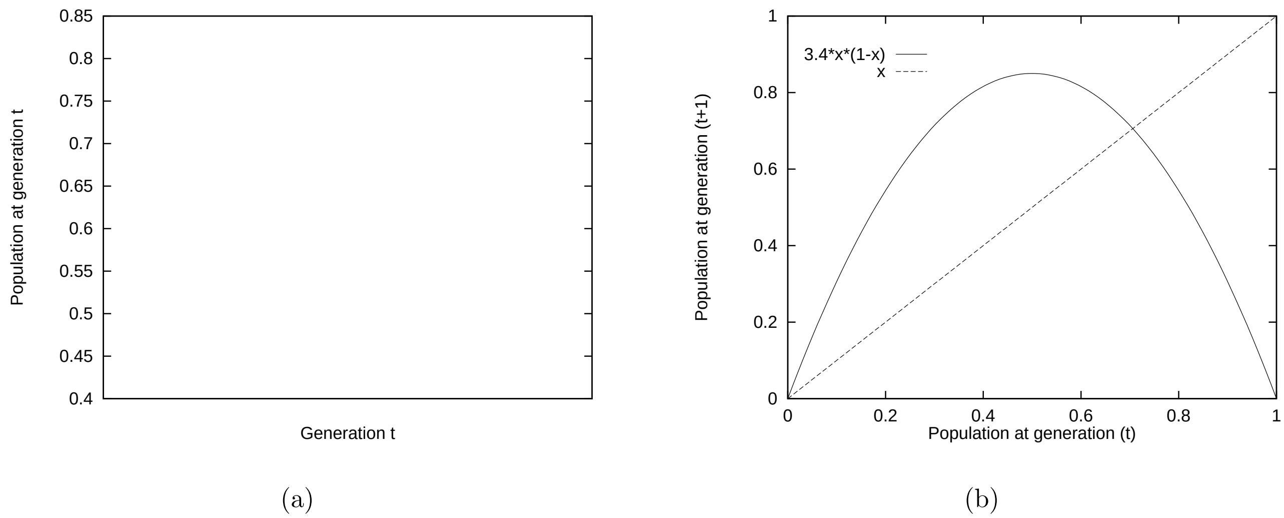

Figure 7: Iteration of the logistic map $x_{t+1} = rx_t(1 - x_t)$, $r = 3.4$, $x_0 = 0.7$, indicating convergence to a period 2 solution.

8 Linear stability analysis of period-2 solutions

In this section we will investigate what happens in the logistic equation when r is slightly larger than 3. The key points of this section are:

1. To understand what is meant by a period 2 solution of a difference equation.
2. To be able to calculate the stability of a period 2 solution.

8.1 The existence of period-2 solutions in the logistic equation

What is the solution of the logistic equation when r passes through the value 3? We know from the previous section that when $3 < r \leq 4$ the fixed points $x^* = 0$ and $x^* = (r-1)/r$ are both unstable. So the solution isn't a period-1 solution. In exploring the behaviour of the logistic map over the range $3 < r \leq 4$ it is useful to introduce the following notation for the iterative procedure:

$$x_1 = f(x_0) \tag{47}$$

$$x_2 = f(x_1) = f(f(x_1)) = f^2(x_0) \tag{48}$$

$$\vdots$$

$$u_n = f(x_{n-1}) = f(f(x_{n-2})) = f^n(x_0) \tag{49}$$

Figure 7 shows an example of iterations of the logistic map, starting from the initial value $x_0 = 0.7$, with the parameter choice $r = 3.4$. The solution is converging to a sequence $p, q, p, q, p, \ldots$, where $f(p) =$ ______ and $f(q) =$ ______. This is a ______ solution. Note that

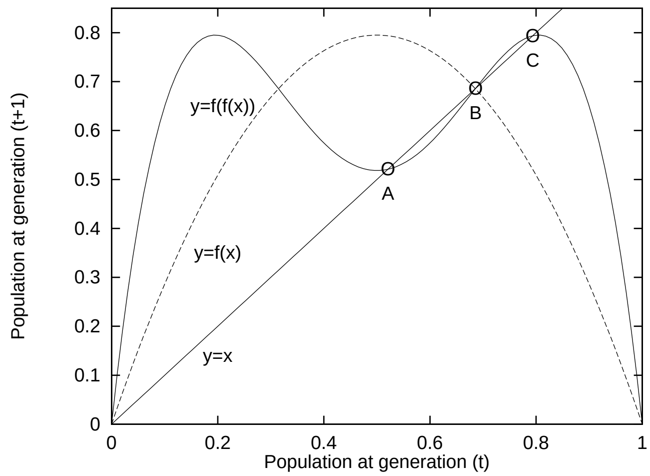

Figure 8: Second iteration $x_{n+2} = f^2(x_n)$ as a function of x_n for the logistic map when $r = 3.18$. The dotted line reproduces the first iteration curve of x_{n+1} as a function of x_n; it passes through the origin and B, the unstable period-1 steady states.

$f^2(p) =$ ______ and $f^2(q) =$ ______. Another way to say this is that p and q are fixed points of the map ______.

Definition 2 (Period-2 solution) *A period-2 solution is a pair* x_0^*, x_1^* *with* $f(x_0^*) = x_1^*$ *and* $f(x_1^*) = x_2^*$ *but* $x_0^* \neq x_1^*$.

For the logistic model (16) we have

$$\begin{aligned} x_{n+1} &= r x_n (1 - x_n) && (50) \\ x_{n+2} &= r x_{n+1} (1 - x_{n+1}) && (51) \\ &= && (52) \end{aligned}$$

We now look at the second iteration (52) and ask if it has any fixed points, i.e. are there any values x_2^* for which $x_{n+2} = x_n = x_2^*$? The function $x_{n+2} = f(x_n)$ is shown in figure 8.

Question 18 *How many fixed points does the map* $x_{n+2} = f^2(x_n)$ *have in figure 8?*

Fixed points of the map $x_{n+2} = f^2(x_n)$ satisfy the equation

$$\begin{aligned} x_2^* &= f^2(x_2^*), && (53) \\ x_2^* &= \qquad , && (54) \\ 0 &= x_2^* \{ r [r (1 - x_2^*)] [1 - r x_2^* (1 - x_2^*)] - 1 \}. && (55) \end{aligned}$$

Fixed points of the map $x_{n+2} = f^2(x_t)$ obviously include fixed points of the map ______. Thus two factors of equation (55) are ______ and ______. Using this knowledge equation (55) can be factored

$$x_2^* [r x_2^* - (r - 1)] \left[r^2 {x_2^*}^2 - r (r + 1) x_2^* + (r + 1) \right] = 0. \qquad (56)$$

Question 19 *Find the solutions to equation (56) when* $r = 3.4$. *Compare your results to figure 7.*

The general solution to equation (56) is

$$x_2^* = 0 \qquad \text{or} \qquad x_2^* = \frac{r-1}{r} > 0 \qquad \text{if} \qquad r > 1, \tag{57}$$

$$x_{2\pm}^* = \qquad \text{if} \qquad r > 3. \tag{58}$$

Thus the equation $x_{n+2} = f^2(x_n)$ has four fixed points if $r > 3$. Two of these are period-1 solutions and two are not. The latter comprise the period-2 solution. In figure 8 B, equal to $(r-1)/r$, is the unstable period-1 solution lying between the new positive equilibria A and C, $x_{2\pm}^*$, in (59) which appear when $r > 3$.

8.1.1 Stability of period 2 orbits

In section 5 we discussed how the *stability* of a fixed point is an important concept which can give us important qualitative information about our system. We also showed how to determine the stability of a fixed point. We will use these ideas to calculate the stability of a period-2 solution.

Suppose that the pair x_0^*, x_1^* is a period-2 orbit for the equation

$$x_{n+1} = f(x_n). \tag{59}$$

The key result in this section is that the eigenvalue of the period-2 solution is given by

$$\lambda = \frac{\mathrm{d}f}{\mathrm{d}x}(x_0^*) \cdot \frac{\mathrm{d}f}{\mathrm{d}x}(x_1^*). \tag{60}$$

The period-2 points x_0^* and x_1^* are fixed points of the map

$$x_{n+2} = f^2(x_n). \tag{61}$$

Thus if we start at x_0^*, we come back to it after two iterations, that is

$$x_2^* = f^2(x_0^*) = x_0^* \qquad \text{but} \qquad x_1^* = f(x_0^*) \neq x_0^*. \tag{62}$$

We shall now apply the stability test, definition 1, to the pair x_0^* and x_1^*. We need the derivative of the second composition map f^2.

Consider the equation

$$F = f(g(x)) \tag{63}$$

Let $u = g(x)$. Then

$$F = f(u), \tag{64}$$

and by the chain rule we have

$$\frac{\mathrm{d}F}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x} \tag{65}$$

Thus

$$\frac{\mathrm{d}F}{\mathrm{d}x} = \tag{66}$$

$$= \tag{67}$$

$$= \tag{68}$$

We are considering the case

$$F = f^2(x), \tag{69}$$

$$= f(f(x)). \tag{70}$$

Question 20 *Compare equations (63) & (70). Identify the function g in equation (70)*

Thus the eigenvalue of the period-2 solution is given by

$$\frac{\mathrm{d}F}{\mathrm{d}x} = \tag{71}$$

Let us calculate the stability of the fixed point x_0^*. The important fact is that $f(x_0^*) =$ ____, so that

$$\frac{\mathrm{d}F}{\mathrm{d}x}(x_0^*) = \quad , \tag{72}$$

$$= \tag{73}$$

How about the the stability of the fixed point x_1^*? The important fact is that $f(x_1^*) =$ ____, so that

$$\frac{\mathrm{d}F}{\mathrm{d}x}(x_1^*) = \quad , \tag{74}$$

$$= \tag{75}$$

Equations (73&75) show that the stability of a period-2 solution is a simultaneous property of the fixed points comprising it. The conclusion is that a period-2 orbit x_0^*, x_1^* is stable if

$$\left|\frac{\mathrm{d}F}{\mathrm{d}x}(x_0^*) \cdot \frac{\mathrm{d}F}{\mathrm{d}x}(x_0^*)\right| \quad , \tag{76}$$

and unstable if

$$\left|\frac{\mathrm{d}F}{\mathrm{d}x}(x_0^*) \cdot \frac{\mathrm{d}F}{\mathrm{d}x}(x_0^*)\right| \quad . \tag{77}$$

8.2 Stability of period-2 solutions in the logistic equation

It is possible to show that the eigenvalue of the period-2 solution in the logistic equation is given by

$$f^{2'}\left(x^*_{2\pm}\right) = 4 + 2r - r^2 \tag{78}$$

Question 21

(a) Sketch equation (78).

(b) For what value of r does $f^{2'}\left(x^*_{2\pm}\right) = 1$*?*

(c) For what value of r does $f^{2'}\left(x^*_{2\pm}\right) = -1$*?*

(d) For what values of r is the period-2 solution stable?

Question 22 (2001 Quiz) *Consider the logistic difference equation*

$$x_{t+1} = r x_t \left(1 - x_t\right).$$

1. *Which (if any) of the following pairs* (x^*_0, x^*_1) *is a period-2 solution of the logistic equation for the given value of r? Explain your answer (calculate all values to 5 decimal places)* [4 *marks*]
   (a) $r = 3.5$, $u^*_0 = 0.88025$, $u^*_1 = 0.36893$.
   (b) $r = 3.7$, $u^*_0 = 0.88025$, $u^*_1 = 0.39002$.
2. *When* $r = 3.9$ *the pair* $x^*_0 = 0.897436$, $x^*_1 = 0.358974$ *is a period-2 solution.*
   (a) *Calculate the eigenvalue of the period-2 solution.* [2 *marks*]
   (b) *Is the period-2 solution stable or unstable? Explain your answer.* [1 *mark*]

Question 23 *Consider the model*

$$x_{t+1} = x_t \exp\left[r\left(1 - x_t\right)\right], \tag{79}$$

which is known as the Ricker model. When $r = 2.2$ *this model as a period-2 solution* $x^*_0 = 0.49706, x^*_1 = 1.50294$.

(a) Show that within 5 decimal places $f\left(x^*_0\right) = x^*_1$ *and* $f\left(x^*_1\right) = x^*_0$.

(b) Calculate the eigenvalue of the period-2 solution.

(c) Is the period-2 solution stable or unstable?

9 Stability of period-n orbits

9.1 Theory

Definition 3 (Composition of a map) *The n-fold composition f_n of the map f is defined inductively by*

$$f^n(x) = f^{n-1}(f(x)). \tag{80}$$

Definition 4 (Periodic Orbit) *A periodic orbit of period n is a sequence $\{x_0^*, \ldots, x_{n-1}^*\}$ for which*

$$x_1^* = f(x_0^*), x_2^* = f(x_1^*), \ldots x_{n-1}^* = f(x_{n-2}^*), x_0^* = f(x_{n-1}^*), \tag{81}$$

with the property that all the points are distinct from each other.

Observe that $x_0 = f^n(x_0)$ so that the point x_0 is a fixed point of the mapping f^n.

The points $\{x_0^*, \ldots, x_{n-1}^*\}$ comprising a period-n orbit are also known as a n-cycle. Obviously each member (x_i^*) of a n-cycle is a fixed point of the mapping f^n. Furthermore, because of the requirement that all the points be distinct, they cannot be fixed points of any composition f^m for which $m < n$.

To calculate the stability of a fixed-point x^* of the mapping f^1 we required the eigenvalue $f'(x^*)$. We now extend this definition to an n-cycle of points $\{x_0^*, \ldots, x_{n-1}^*\}$.

To do this we need a general formula for the derivative of an n-fold composition f_n. It is quite simple.

Definition 5 (Eigenvalue of a n-cycle) *Suppose that $x_0, x_1, \ldots, x_{n-1}$ is an n-cycle. Then*

$$(f^n)'(x_i) = \tag{82}$$

Thus the eigenvalue of a n-cycle is the product of derivatives of f at each of the n successive places visited, starting from x_i.

It follows from the formula for $(f^n)'(x)$ that points on a periodic orbit all have the same stability, since the values of the derivatives $f'(x_i^*)$ which go into the calculation of $(f^n)'(x_i^*)$ are evaluated at precisely every point of the periodic orbit, each once and only once. This is independent of where we start on the orbit, because it is periodic. So we have a simple derivative test to determine the stability of a periodic orbit.

Definition 6 (Stability of a periodic orbit) *A periodic orbit is stable, or unstable, according as the product of the derivatives $f'(x_i^*)$ taken over the orbit has magnitude less than, or greater than, unity.*