On circulant best matrices and their applications

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Abstract

Call four type 1 $(1, -1)$ matrices, $X_1, X_2, X_3, X_4$, of the same group of order $m$ (odd) with the properties (i) $(X_i - I)^T = -(X_i - I)$, $i = 1, 2, 3$, (ii) $X_i^T = X_4$ and the diagonal elements are positive, (iii) $X_i X_j = X_j X_i$ and (iv) $X_1 X_1^T + X_2 X_2^T + X_3 X_3^T + X_4 X_4^T = 4mI_m$, best matrices. We use a computer to give, for the first time, all inequivalent best matrices of odd order $m \leq 31$. Inequivalent best matrices of order $m$, $m$ odd, can be used to find inequivalent skew-Hadamard matrices of order $4m$. We use best matrices of order $\frac{1}{4}(s^2 + 3)$ to construct new orthogonal designs, including new $OD(2s^2 + 6; 1, 1, 2, 2, s^2, s^2)$. AMS Subject Classification: Primary 05B20, Secondary 05B30

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1 Introduction and basic definitions

A $(1, -1)$ matrix of order $n$ is called a Hadamard matrix if $HH^T = H^TH = nI_n$, where $H^T$ is the transpose of $H$ and $I_n$ is the identity matrix of order $n$. A $(1, -1)$ matrix $A$ of order $n$ is said to be of skew type if $A - I_n$ is skew-symmetric. If $A$ is a skew type Hadamard matrix then $A$ is said to be a skew-Hadamard matrix. Two $(1, -1)$ matrices $A, B$ of order $n$ are said to be amicable if $AB^T = BA^T$.

Let $G$ be an additive abelian group of order $n$ with elements $g_1, g_2, \ldots, g_n$ and $X$ a subset of $G$. Define the type 1 $(1, -1)$ incidence matrix $M = (m_{ij})$ of order $n$ of $X$ to be

$$m_{ij} = \begin{cases} +1 & \text{if } g_i - g_j \in X \\ -1 & \text{otherwise} \end{cases}$$

and the type 2 $(1, -1)$ incidence matrix $N = (n_{ij})$ of order $n$ of $X$ to be

$$n_{ij} = \begin{cases} +1 & \text{if } g_i + g_j \in X \\ -1 & \text{otherwise} \end{cases}$$

In particular, if $G$ is cyclic the matrices $M$ and $N$ are called circulant and back circulant respectively. In this case $m_{ij} = m_{1, j-i+1}$ and $n_{ij} = n_{1, i+j-1}$ respectively (indices should be reduced modulo $n$).

Definition 1 Let $X_1, X_2, X_3, X_4$ be four type 1 $(1, -1)$ matrices on the same group of order $m$ (odd) with the properties

(i) $(X_i - I)^T = -(X_i - I)$, $i = 1, 2, 3$

(ii) $X_4^T = X_4$ and the diagonal elements are positive

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(iii) \( X_iX_j = X_jX_i \)

(iv) \( X_1X_1^T + X_2X_2^T + X_3X_3^T + X_4X_4^T = 4mI_m \)

Call such matrices best matrices of order \( m \).

Pre and post multiplying equation (iv) by \( e \) and \( e^T \), respectively, where \( e \) is the \( 1 \times m \) matrix of all ones gives that \( 4m - 3 = s^2 \) where \( s \) is odd integer.

In this paper we consider circulant best matrices, so condition (iii) is trivially satisfied. Hence, multiplying on the left by \( e^T \) (the \( 1 \times m \) vector of one's) and on the right by \( e \) both sides of (iv) we conclude that circulant (or type 1) best matrices can only exist for orders \( m \) of which \( 4m = 1^2 + 1^2 + 1^2 + a^2 \), where \( a \) is the sum of the elements of the first row of the symmetric matrix \( X_4 \) and \( a \) is an odd integer.

An orthogonal design of order \( n \) and type \( (s_1, s_2, \ldots, s_u) \) (\( s_i > 0 \)), denoted \( OD(n; s_1, s_2, \ldots, s_u) \), on the commuting variables \( x_1, x_2, \ldots, x_u \), is an \( n \times n \) matrix \( A \) with entries from \( \{0, \pm x_1, \pm x_2, \ldots, \pm x_u\} \) such that

\[
AA^T = \left( \sum_{i=1}^{u} s_i x_i^2 \right) I_n
\]

Alternatively, the rows of \( A \) are formally orthogonal and each row has precisely \( s_i \) entries of the type \( \pm x_i \). In [1], where this was first defined, it was mentioned that

\[
A^TA = \left( \sum_{i=1}^{u} s_i x_i^2 \right) I_n
\]

and so our alternative description of \( A \) applies equally well to the columns of \( A \). It was also shown in [1] that \( u \leq \rho(n) \), where \( \rho(n) \) (Radon’s function) is defined by \( \rho(n) = 8c + 2d \), when \( n = 2^b \cdot b \) odd, \( a = 4c + d \), \( 0 \leq d < 4 \). For more details and constructions of orthogonal designs the reader can consult the book of Geramita and Seberry [2].

In section 2 we describe briefly the method of construction, in section 3 we give all inequivalent circulant best matrices of odd order \( m \leq 31 \), and in section 4 we use best matrices to construct some new orthogonal designs and families of Hadamard matrices.

## 2 Method of construction

In order to describe our construction for best matrices, we need a few more definitions. Let \( n \) be a positive integer.

**Definition 2** Four subsets \( S_0, S_1, S_2, S_3 \) of \( \{1, 2, \ldots, n-1\} \) are called \( 4-(n; n_0, n_1, n_2, n_3; \lambda) \) supplementary difference sets (sds) modulo \( n \) if \( |S_k| = n_k \) for \( k = 0, 1, 2, 3 \) and for each \( m \in \{1, 2, \ldots, n-1\} \) we have \( \lambda_0(m) + \lambda_1(m) + \lambda_2(m) + \lambda_3(m) = \lambda \), where \( \lambda_k(m) \) is the number of solutions \( (i, j) \) of the congruence \( i - j \equiv m \) (mod \( n \)) with \( i, j \in S_k \).

Suppose that \( S_k \) are \( 4-(n; n_0, n_1, n_2, n_3; \lambda) \) sds modulo \( n \) having the following additional properties:

\[
\begin{align*}
 n + \lambda &= n_0 + n_1 + n_2 + n_3 \quad \text{(1)} \\
i \in S_k &\iff n - i \not\in S_k, \quad k = 0, 1, 2 \quad \text{(2)} \\
i \in S_t &\iff n - i \in S_t, \quad t = 3 \quad \text{(3)}
\end{align*}
\]
where in (2) and (3) it is assumed that \( i \in \{1, 2, \ldots, n - 1\} \).

Let \( a_k = (a_k, a_{k+1}, \ldots, a_{k+n-1}) \), \( k = 0, 1, 2, 3 \), be the row vector defined by
\[
a_k = \begin{cases} 
-1 & \text{if } i \in S_k \\
1 & \text{otherwise}
\end{cases}
\]

Furthermore let \( A_k, k = 0, 1, 2, 3 \) be the circulant matrices with first row \( a_k \). Then it can be easily verified that \( A_0, A_1, A_2, A_3 \) are four matrices of order \( n \) as described in definition 1.

Let \( r \) be an integer relatively prime to \( n \), and set
\[ S'_k = \{ r \, \text{(mod } n) : i \in S_k \} \subset \{1, 2, \ldots, n - 1\} \]
for \( k = 0, 1, 2, 3 \). These sets are also \( 4 - (n; n_0, n_1, n_2, n_3; \lambda) \) sds modulo \( n \) satisfying the conditions (1), (2), (3). We shall say that such quadruples \( S_0, S_1, S_2, S_3 \) and \( S'_0, S'_1, S'_2, S'_3 \) are equivalent.

We now give a brief description of the method of computation used to find the necessary sds’s. The numbers \( n_i \) are easy to determine (see [6]). We first generate a number of subsets of size \( n_i \) of \( \{1, 2, \ldots, n\} \) having the required symmetry properties (2) or (3), and at the same time compute the corresponding set of differences. We store the multiplicities of these differences in a file, say \( f_1 \), saving only sets of differences with different multiplicities. After creating these files for each of the sizes \( n_0, \ldots, n_3 \), we try to match the items in the four files to produce an sds. This is done by examining items in two files only, say \( f_0 \) and \( f_1 \) and creating a new file in which we record the pairs which produce different total multiplicities of the differences. The procedure is repeated with the remaining two files \( f_2 \) and \( f_3 \). Finally the resulting two files are examined in order to find a perfect match.

The results that we found applying this algorithm are presented in the next section.

3 The inequivalent supplementary difference sets

In this section we give for the first time all inequivalent supplementary difference sets which satisfy the condition (1), (2), (3) for all odd \( m \leq 31 \).

<table>
<thead>
<tr>
<th>Table 1.</th>
<th>All inequivalent circulant best matrices of odd order ( m \leq 31 )</th>
</tr>
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<tbody>
<tr>
<td><strong>m = 3</strong>; ( 4 - (3; 1, 1, 0; 0) )</td>
<td>( S_0 = {1}, \ S_1 = {1}, \ S_2 = {1}, \ S_3 = \emptyset )</td>
</tr>
<tr>
<td><strong>m = 7</strong>; ( 4 - (7; 3, 3, 6; 8) )</td>
<td>( S_0 = {1, 3, 5}, \ S_1 = {1, 2, 3}, \ S_2 = {1, 4, 5}, \ S_3 = {1, 2, 3, 4, 5, 6} )</td>
</tr>
<tr>
<td><strong>m = 13</strong>; ( 4 - (13; 6, 6, 10; 15) )</td>
<td>( S_0 = {1, 3, 5, 6, 9, 11}, \ S_1 = {2, 6, 8, 9, 10, 12}, \ S_2 = {1, 4, 5, 6, 10, 11}, \ S_3 = {2, 3, 4, 5, 6, 7, 8, 9, 10, 11} )</td>
</tr>
</tbody>
</table>
2. \( S_0 = \{2, 6, 89, 10, 12\}, S_1 = \{1, 4, 5, 6, 10, 11\}, \\
S_2 = \{1, 2, 3, 4, 6, 8\}, S_3 = \{1, 2, 3, 4, 6, 7, 9, 10, 11, 12\} \)

Table 1. (continued)

\[ m = 21; \ 4 - (21; 10, 10, 10, 6; 15) \]

1. \( S_0 = \{1, 3, 6, 10, 12, 13, 14, 16, 17, 19\}, S_1 = \{1, 2, 3, 4, 7, 8, 10, 12, 15, 16\}, \\
S_2 = \{1, 2, 5, 11, 12, 13, 14, 15, 17, 18\}, S_3 = \{4, 6, 10, 11, 15, 17\}, \)

2. \( S_0 = \{1, 3, 4, 5, 6, 8, 10, 12, 14, 19\}, S_1 = \{1, 2, 3, 4, 7, 9, 10, 13, 15, 16\}, \\
S_2 = \{3, 4, 6, 7, 8, 9, 11, 16, 19, 20\}, S_3 = \{2, 8, 9, 12, 13, 19\}, \)

3. \( S_0 = \{1, 3, 4, 6, 10, 12, 13, 14, 16, 19\}, S_1 = \{2, 3, 4, 7, 8, 9, 10, 15, 16, 20\}, \\
S_2 = \{1, 2, 3, 4, 5, 7, 8, 11, 12, 15\}, \quad S_3 = \{1, 3, 8, 13, 18, 20\}, \)

4. \( S_0 = \{1, 2, 4, 6, 7, 9, 10, 13, 16, 18\}, S_1 = \{4, 5, 10, 12, 13, 14, 15, 18, 19, 20\}, \\
S_2 = \{1, 2, 3, 4, 5, 6, 9, 11, 13, 14\}, \quad S_3 = \{1, 5, 9, 12, 16, 20\}, \)

5. \( S_0 = \{1, 2, 4, 6, 7, 9, 10, 13, 16, 18\}, \quad S_1 = \{1, 2, 3, 4, 8, 11, 12, 14, 15, 16\}, \\
S_2 = \{1, 6, 11, 12, 13, 14, 16, 17, 18, 19\}, S_3 = \{2, 6, 8, 13, 15, 19\}, \)

6. \( S_0 = \{1, 3, 7, 10, 12, 13, 15, 16, 17, 19\}, S_1 = \{1, 2, 3, 4, 6, 7, 8, 11, 12, 16\}, \\
S_2 = \{1, 3, 4, 5, 6, 9, 12, 13, 14, 19\}, \quad S_3 = \{1, 5, 9, 12, 16, 20\}, \)

7. \( S_0 = \{1, 2, 4, 6, 7, 9, 10, 13, 16, 18\}, \quad S_1 = \{1, 2, 3, 4, 6, 7, 8, 11, 12, 16\}, \\
S_2 = \{1, 7, 8, 9, 11, 15, 16, 17, 18, 19\}, S_3 = \{2, 4, 9, 12, 17, 19\}, \)

\[ m = 31; \ 4 - (31; 15, 15, 15, 10; 24) \]

\( S_0 = \{1, 5, 6, 8, 10, 13, 14, 16, 19, 20, 22, 24, 27, 28, 29\}, \\
S_1 = \{2, 3, 4, 5, 6, 7, 10, 12, 13, 14, 16, 20, 22, 23, 30\}, \\
S_2 = \{1, 3, 4, 5, 12, 16, 17, 18, 20, 21, 22, 23, 24, 25, 29\}, \\
S_3 = \{3, 6, 8, 12, 13, 18, 19, 23, 25, 28\}, \)

\( S_0 = \{1, 4, 7, 10, 12, 14, 15, 18, 20, 22, 23, 25, 26, 28, 29\}, \\
S_1 = \{1, 2, 6, 12, 13, 14, 16, 20, 21, 22, 23, 24, 26, 27, 28\}, \\
S_2 = \{5, 6, 7, 9, 10, 11, 12, 14, 15, 18, 23, 27, 28, 29, 30\}, \\
S_3 = \{2, 3, 10, 12, 14, 17, 19, 21, 28, 29\}, \)

4 Constructions using best matrices

**Theorem 1** Suppose there exist best matrices of order \( t = \frac{1}{4}(s^2 + 3) \), then there exists an \( OD(8t; 1, 1, 2, 2, s^2, s^2) \).

**Proof.** Suppose \( I + X, I + X_2, I + X_3 \) and \( X_4 \) are the circulant best matrices of order \( t \). Let \( a, b, c, d, e, f \) be commuting variables. Define

\[
X = \frac{1}{2}(X_2 + X_3); \quad Y = \frac{1}{2}(X_2 - X_3)
\]

So

\[
X^T = -X, \quad Y^T = -Y, \quad XX^T + YY^T = \frac{1}{2}(X_2X_2^T + X_3X_3^T).
\]

Now define
\[
A_1 = aI + bX_1
\]
\[
A_2 = dX_4
\]
\[
A_3 = eI - dX_1
\]
\[
A_4 = bX_4
\]
\[
A_5 = eI + bX + dY
\]
\[
A_6 = fI - dX + bY
\]
\[
A_7 = eI - bX - dY
\]
\[
A_8 = fI + dX - bY
\]

It is straightforward to check, using the properties of best matrices, that \(A_1, A_2, \ldots, A_8\) satisfy the additive property and

\[
\sum_{i=1}^{8} A_i A_i^T = (a^2 + c^2 + 2e^2 + 2f^2 + (4t - 3)b^2 + (4t - 3)d^2)I_t.
\]

We now check if the matrices form an amicable set. First we see

\[
A_1 A_2^T = adX_4 + bdX_1 X_4
\]
\[
A_2 A_1^T = adX_4 - bdX_1 X_4
\]
\[
A_3 A_4^T = cbX_4 - bdX_1 X_4
\]
\[
A_4 A_3^T = cbX_4 + bdX_1 X_4
\]

So

\[
A_1 A_2^T - A_2 A_1^T + A_3 A_4^T - A_4 A_3^T = 0.
\]

Then we have

\[
A_5 A_6^T = efI + bfX + dfY + edX + bdX^2 + d^2XY - ebY - b^2XY - dbY^2
\]
\[
A_6 A_5^T = efI - bfX - dfY - edX + bdX^2 + d^2XY + ebY - b^2XY - dbY^2
\]
\[
A_7 A_8^T = efI - bfX - dfY - edX + bdX^2 + d^2XY + ebY - b^2XY - dbY^2
\]
\[
A_8 A_7^T = efI + bfX + dfY + edX + bdX^2 + d^2XY - ebY - b^2XY - dbY^2
\]

So

\[
A_5 A_6^T - A_6 A_5^T + A_7 A_8^T - A_8 A_7^T = 0.
\]

Hence \(A_1 \ldots A_8\) are amicable set of circulant matrices satisfying the additive property. Hence we may use them in Kharaghani array \([3]\) to form \(OD(8t; 1, 1, 2, 2, 4t - 3, 4t - 3)\). \(\square\)

**Remark 1** We note there is no construction known which gives \(OD(8t; 1, 1, 2, 2, 4t - 3, 4t - 3)\).

Hence we have \(OD(56; 1, 1, 2, 2, 25, 25), OD(104; 1, 1, 2, 2, 49, 49), OD(168; 1, 1, 2, 2, 81, 81)\) and \(OD(248; 1, 1, 2, 2, 121, 121)\) for the first time.

**Theorem 2** Suppose there are best matrices of order \(m\) then there exists an \(OD(4m; 1, 1, 1, 4m - 3)\).
Proof. Let $x_1, x_2, x_3$ and $x_4$ be four commuting variables. Write $I + B_1, I + B_2, I + B_3$ and $B_4$ for the best matrices of order $m$. Further write $A_1 = x_1 I + x_2 B_1, A_2 = x_2 I + x_4 B_2, A_3 = x_3 I + x_4 B_3$ and $A_4 = x_4 B_4$ for the four circulant (or type 1) matrices of order $m$ satisfying

$$A_1A_1^T + A_2A_2^T + A_3A_3^T + A_4A_4^T = (x_1^2 + x_2^2 + x_3^2 + (4m - 3)x_4^2)I_m.$$ 

Let $R = r_{ij}$, where $r_{ij} = 1$ for $i + j = m + 1$ and 0 otherwise. Then using the Goethals-Seidel array

$$
\begin{bmatrix}
    A_1 & A_2 R & A_3 R & A_4 R \\
    -A_2 R & A_1 & A_4^T R & -A_3^T R \\
    -A_3 R & -A_4^T R & A_1 & A_2^T R \\
    -A_4 R & A_3^T R & -A_2^T R & A_1
\end{bmatrix},
$$

is the required $OD(4m; 1, 1, 1, 4m - 3)$. \hfill \Box

Corollary 2 Let $m$ be the order of best matrices. Then an $OD(4m; 1, 1, 1, 4m - 3)$ exists.

Corollary 3 Let $m \in \{3, 7, 13, 21, 31\}$. Then an $OD(4m; 1, 1, 1, 4m - 3)$ exists.

Corollary 4 Let $m$ be the order of best matrices. Then there exist up to 8 inequivalent skew-Hadamard, and Hadamard, matrices of order $4m$.

Proof. Let $X_1, X_2, X_3, X_4$ be best matrices of order $m$. Then choosing $A_1 = X_1, A_2 = I \pm (X_2 - I), A_3 = I \pm (X_3 - I)$ and $A_4 = \pm X_4$, in the Goethals-Seidel array gives the required result, (Note choosing $A_2 = \pm I + (X_2 - I)$, and $A_3 = \pm I + (X_3 - I)$ is an alternative choice.) \hfill \Box

We have constructed the Hadamard matrices of order 28 made, using as $A_1, A_2, A_3$ and $A_4$, the first rows given below in the Goethals-Seidel array

| $\begin{array}{cccccc}
1 & 1 & 1 & -1 & 1 & -1-1; \\
1 & 1 & 1 & -1 & 1 & 1-1-1; \\
1 & 1 & 1 & -1 & 1 & 1-1-1; \\
1 & 1 & 1 & -1 & 1 & 1-1-1;
\end{array}$ | $\begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & 1-1-1; \\
1 & 1 & 1 & 1 & 1 & 1-1-1; \\
1 & 1 & 1 & 1 & 1 & 1-1-1; \\
1 & 1 & 1 & 1 & 1 & 1-1-1;
\end{array}$ |

We believe that the four Hadamard matrices thus produced are H-inequivalent and inequivalent skew-Hadamard matrices.

Corollary 5 Suppose there are best matrices of order $m$ and an Hadamard matrix, $H$, of order $4m/3$, then there is an Hadamard matrix of order $4m(4m - 3)/3$.

Proof. Use the best matrices to make an $OD(4m; 1, 1, 1, 4m - 3)$.

Write $J$ for the best matrices to make an $OD(4m; 1, 1, 1, 4m - 3)$ of all ones. Normalize the Hadamard matrix, $H$, of order $4m/3$ so that its first row and column is all ones, then discard the first row and column to obtain the core of the Hadamard matrix, $B$, of order $4m/3 - 1$, which satisfies $BJ = -J$ and $BB^T = 4m/3I_{4m/3-1} = J_{4m/3-1}$. Then replacing the variables of the $OD(4m; 1, 1, 1, 4m - 3)$ by $J$, $J$, $J$ and $B$, which satisfy

$$3JJ^T + (4m - 3)BB^T = (4m - 3)J + 4m(4m - 3)/3I - (4m - 3)J = 4m(4m - 3)/3I,$$

gives the required matrix. \hfill \Box
Example 1 We have found best matrices of orders $m = 3$ and 21. These give Hadamard matrices of orders 36 and 2268. These orders are not new, but, since Kimura [4, 5] has found some 487 inequivalent Hadamard matrices of order 28 which can be used in the corollary for $m = 21$ we may have constructed new, inequivalent, Hadamard matrices of order 2268. Since the variables can also be replaced by $J$, $± J$, $± J$ and $± B$ there is further potential for inequivalent Hadamard matrices.

Corollary 6 Suppose there are best matrices of order $m$ and a symmetric Hadamard matrix of order $h$

1. $h = 4(m + 1)/3$;
2. $h = 4(m + 2)/3$;
3. $h = 4(m + 3)/3$,

then there is an Hadamard matrix of order $4m(h - 1)$.

Proof. Use the best matrices to make an $OD(4m; 1, 1, 1, 4m - 3)$.

Normalize the symmetric Hadamard matrix of order $h$ so that its first row and column is all ones, then discard the first row and column to obtain the symmetric core of the symmetric Hadamard matrix, $B$, which satisfies $BJ = - J$ and $BB^T = hI_{h-1} - J_{h-1}$. Write $K = J - 2I$. Then

$$KJ^T = JK^T; \quad KB^T = BK^T; \quad JB^T = BJ^T.$$ 

Then replacing the variables of the $OD(4m; 1, 1, 1, 4m - 3)$ by

1. $J$, $J$, $K$ and $B$;
2. $J$, $K$, $K$ and $B$;

which satisfy

$$2J^T + K^T + (4m - 3)BB^T = 2(h-1)J + (h-5)J + 4I + h(4m-3)I - (4m-3)J = 4m(h-1)I;$$

$$J^T + 2K^T + (4m - 3)BB^T = (h-1)J + 2(h-5)J + 8I + h(4m-3)I - (4m-3)J = 4m(h-1)I;$$

$$3K^T + (4m - 3)BB^T = 3(h-5)J + 12I + h(4m-3)I - (4m-3)J = 4m(h-1)I,$$

respectively giving the required matrices.

Example 2 From above we have four sequences of lengths $m = 3$, 7, 13, 21 and 31 which are the first rows for best matrices. Then using Corollary 5 and the best matrices of orders 3 and 21 we obtain Hadamard matrices of order 36 and 2268. Using Corollary 6 we obtain Hadamard matrices of orders $84 = 4 \cdot 21$, 308 = $4 \cdot 77$, 988 = $4 \cdot 247 = 4 \cdot 13 \cdot 19$, 2604 = $4 \cdot 851 = 4 \cdot 21 \cdot 31$ and 5332 = $4 \cdot 31 \cdot 43$. None of these orders are new but there are possibly inequivalent Hadamard matrices.

Corollary 7 Suppose there are best matrices of order $m$, a back-circulant $SBIBD(v, k, \lambda)$ and an Hadamard matrix with circulant core, $B$, of order
1. \( v = 4(k - \lambda) + 4m/3 - 1; \)

2. \( v = (8k - 8\lambda + 4m)/3 - 1; \)

3. \( v = 4(k - \lambda + m)/3 - 1; \)

then there is an Hadamard matrix of order \( 4mv. \)

**Proof.** Form the \( OD(4m; 1, 1, 1, 4m - 3) \) as before.

As before \( B \) satisfies \( BJ = -J \) and \( BB^T = (v + 1)I_v - J_v. \) Let \( A \) be the \( \pm 1 \) incidence matrix of the \( SBI B D(v, k, \lambda) \) then \( AJ = (2k - v)J \) and \( AA^T = 4(k - \lambda)I + (v - 4(k - \lambda))J. \) We note \( AB^T = BA^T \) as \( A \) is back-circulant and \( B \) is circulant. We now replace the variables of the \( OD(4m; 1, 1, 1, 4m - 3) \) by (1) \( A, A, A \) and \( B, \) (2) \( A, A, J \) and \( B, \) and (3) \( A, J, J \) and \( B, \) respectively, which satisfy

\[
3AA^T + (4m - 3)BB^T = 12(k - \lambda)I + 3(v - 4(k - \lambda))J + (4m - 3)(v + 1)I - (4m - 3)J = 4mvI,
\]

\[
2AA^T + JJ^T + (4m - 3)BB^T = 8(k - \lambda)I + 2(v - 4(k - \lambda))J + vJ + (4m - 3)(v + 1)I - (4m - 3)J = 4mvI,
\]

\[
AA^T + 2JJ^T + (4m - 3)BB^T = 4(k - \lambda)I + (v - 4(k - \lambda))J + 2vJ + (4m - 3)(v + 1)I - (4m - 3)J = 4mvI,
\]

gives the required matrices. \( \square \)

**References**


